

On single-photon wave function

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Abstract

We present in this paper how the single-photon wave function for transversal photons (with the direct sum of ordinary unitary representations of helicity 1 and -1 acting on it) is subsumed within the formalism of Gupta-Bleuler for the quantized free electromagnetic field in the Krein space (i.e. in the ordinary Hilbert space endowed with the Gupta-Bleuler operator η). Rigorous Gupta-Bleuler quantization of the free electromagnetic field is based on a generalization of ours (published formerly) of the Mackey theory of induced representations which includes representations preserving the indefinite Krein inner-product given by the Gupta-Bleuler operator and acting in the Krein space. The free electromagnetic field is constructed by application of the direct sum of (symmetrized) tensor products of a specific indecomposable (but reducible) single-photon representation which is Krein-isometric but non unitary, we call it Łopuszański representation, i. e. we construct the field by application of the Segal's second quantization functor to the specific Krein-isometric representation. A closed subspace \mathcal{H}_{tr} of the single-photon Krein space on which the indefinite Krein-inner-product is strictly positive is constructed such that the Krein-isometric single-photon representation generates modulo unphysical states precisely the action of a representation which preserves the positive inner product on \mathcal{H}_{tr} induced by the Krein inner product, and is equal to the direct sum of ordinary unitary representations of helicity 1 and -1 respectively. Two states of single photon Krein space are physically equivalent whenever differ by a state of Krein norm zero and whose projection on \mathcal{H}_{tr} , in the sense of the Krein-inner-product, vanishes. In particular it follows that the results of Białynicki-Birula on the single-photon wave function may be reconciled with the micro-local perturbative approach to QED initiated by Stückelberg and Bogoliubov.

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1 Introduction

The wave function of single photon in the position picture is a concept which is accompanied with controversial opinions. For example in [8] or [29] there is even stated that position wave function for photon does not exist. We agree e.g. with [2] and [3] and the authors cited there, that quantum field theory speaks for the contrary: generally a free quantum field is constructed by the application of the symmetrized/antisymmetrized tensoring and direct sum operations (the so called second quantized functor) to a specific representation of the double covering of the Poincaré group acting in a space, which may be identified with the space of single particle wave functions, and which depends on the specific quantum field. At the level of the free electromagnetic field one can start at the Hilbert space of transversal single photon states acted on by the direct sum of the unitary zero mass helicity 1 and -1 representations respectively (in the language of the classical by now Wigner-Mackey-Gelfand-Bargmann classification of irreducible unitary representations of the Poincaré group). In more physical terms the representation has been described e.g in [3] together with its relation to the Riemann-Silbertstein vector wave function, and e.g. in [1] in the form more closely related to the Wigner-Mackey scheme. It is true that the (free) quantum electromagnetic field has its own peculiarities making some differences in comparison to massive and non gauge fields. The first peculiarity of a zero mass quantum (free) field, even non gauge field (as we assume for a while in order to simplify situation), is that now the representation of the Poincaré group to which we apply Segal's functor of second quantization although being unitary in ordinary sense, is specified within the Wigner-Mackey classification scheme by the orbit in the momentum space which is the light cone (without the apex), contrary to the massive case, where the orbit is the smooth sheet of the two-sheeted hyperboloid. The apex being a singular point of the cone (in the sense of the ordinary differential structure of the cone as embedded into the \mathbb{R}^4 -manifold) causes serious difficulties of infra-red character. This is because the quantum field is in fact an operator-valued distribution (as motivated by the famous Bohr-Rosenfeld analysis [9] of the measurement of the quantum electromagnetic field) which needs a test function space. It is customary to use the standard Schwartz space of rapidly decreasing functions as the universal test space even for zero mass fields, and this is not the correct test space for zero mass field. Recall that the construction mentioned to above of a free quantum field achieved by the second quantization functor Γ applied to a representation specified by a fixed orbit allows to construct creation and annihilation families of *operators* in the Fock space. In order to construct operator valued fields we have to proceed slightly further along the construction given by Streater and Wightman in their well known monograph [33], Ch. 3. In the construction of Wightman we consider the restrictions of Fourier transforms (i.e. functions in the momentum space) of the test functions to the orbit in question. The construction works if the restriction is a continuous map from the test function space in \mathbb{R}^4 to the test function space in \mathbb{R}^3 which is really the case in the massive case as the the orbit is a smooth manifold in that case.

Unfortunately it seems that it has escaped due attention of physicists that the correct test function space in the momentum representation for the zero mass field is the closed subspace \mathcal{S}_0 of the Schwartz space \mathcal{S} of those functions which vanish at zero together with all their derivatives and the test function space \mathcal{S}_{00} in the position representation is given by the inverse Fourier image of the space \mathcal{S}_0 . This in turn causes additional difficulties concerned with exploring and correct use of the principles of locality character, because in particular the space \mathcal{S}_{00} does not contain any function of compact support (except the trivial zero function) which immediately follows from the generalized Paley-Wiener theorem. Fields have to be carefully extended outside the test space \mathcal{S}_0 on functions whose derivatives *up to only a finite order* vanish at zero (similarly as the creation and annihilation families of operators, which make sense outside the test space) in order to account for the locality-type principles correctly in the zero mass case. The situation for the electromagnetic field is still more delicate as the field is accompanied by the gauge freedom and the ordinary unitarity is untenable and has to be replaced with a weaker condition of preservation of the indefinite Krein-inner product – which is the second main peculiarity of the electromagnetic field, shared with the other zero mass gauge fields of the standard model.

Namely, although we may construct (remembering that we have to be careful with the choice of the test function space) the free quantum electric and magnetic fields by the mentioned application of the Segal second quantization functor to the direct sum of zero mass helicity 1 and -1 unitary representations acting on the Riemann-Silberstein vector function (as described in [3] or [1]), we encounter in this way a difficulty if we would like to restore the connection to the quantum vector potential and its local transformation law within the scheme. In principle we may reconstruct the quantum vector potential in the momentum picture quite easily, but in connection to the non-local relationship of the potential to the electric and the magnetic fields in the position picture the local character of the vector potential is lost. We regard this a weakness when passing to interacting fields, specifically in passing to perturbative QED, and let us explain shortly why this is so. After half a century the causal method of Stückelberg and Bogoliubov turned up to be very valuable in avoiding the ultraviolet divergences in perturbative QFT, compare [15]. Their method have been extended on QED and the other gauge fields, compare [4], [10], [11], [12], [13]. A crucial point of the method is the locality principle (local dependence of the interacting fields on the interaction Lagrangian, [14]), and the second circumstance is that we need to have the quantum vector potential – recall that the minimal coupling is expressed immediately with the vector potential. Joining this prerequisites together we see that we need the vector potential with its local transformation law retained. In particular we can achieve this within the Lorentz gauge with the four vector character of the transformation law of the quantum vector potential. However unitarity will have to be abandoned (recall the Gupta-Bleuler quantization [5]). Indeed, it is known that the four vector transformation law together with the zero mass character of the field cannot be retained together with unitarity of the representation in the single

particle space, compare e.g. [19], [20]. Because we prefer to stay within the micro-local perturbation scheme of QED and other gauge fields of the standard model, avoiding ultraviolet divergences, we choose to abandon unitarity of the representation in the single photon states and replace Hilbert space and unitarity with Krein space and Krein-isometry property of the representation. I.e. we now have an ordinary Hilbert space together with two orthogonal projections P_+ and P_- (of infinite dimensional ranges in our case) summing up to unity: $P_+ + P_- = I$, together with the fundamental symmetry $\mathfrak{J} = P_+ - P_-$ which in case of the Krein space of the free quantum electromagnetic field is called the Gupta-Bleuler operator η . Exactly as the ordinary Hilbert space structure and unitary representation admits the operation of direct sum and tensoring also the Krein space structure and Krein isometric representation preserving the Krein inner product $(\cdot, \mathfrak{J}\cdot)$ (where (\cdot, \cdot) is the ordinary Hilbert space inner product) admits direct summation and tensoring. In order to work effectively with such Krein-isometric representations we need to build a theory which plays the role analogous to the Mackey theory of induced representations. We have constructed such a theory in [34] and in particular we have proved the main theorems, namely we proved the Kronecker product theorem, subgroup theorem and the imprimitivity system theorem to hold for Krein-isometric representations induced by Krein-unitary¹ representations (additional analytic assumptions are sufficiently weak to be effective for physical applications). As an application we construct the free quantum electromagnetic field using the symmetrized tensoring and direct summation to a specific single photon indecomposable (but reducible) Krein-isometric representation of the double covering of the Poincaré group, which we call *Lopuszański representation*. We call the representation with Lopuszański's name because he was the theoretician who suggested generalization of the Mackey theory in Krein spaces and strongly advocated the idea of construction of quantum electromagnetic field based on the representation theory in Krein spaces, compare e.g. [19], [20]. In short we apply the second quantization functor Γ to the *Lopuszański representation* in order to construct the free electromagnetic field. In particular the operator $\eta = \Gamma(\mathfrak{J})$, where \mathfrak{J} is the fundamental symmetry of the single photon representation (i.e. *Lopuszański representation*), is indeed equal to the Gupta-Bleuler operator.

It is our aim of the paper to show that the construction of the transversal single photon space \mathcal{H}_{tr} acted on by the direct sum $[0, 1] \oplus [0, -1]$ of unitary zero mass, helicity 1 and of helicity -1 representation as described in [3] may be reconstructed as the closed subspace of physical states of the single photon Krein space with the representation $[0, 1] \oplus [0, -1]$ induced by the action modulo unphysical states of the single photon Krein-isometric representation. In this sense we extend the results on single photon wave function, e.g. the uncertainty

¹Krein-unitary operator is defined as a bounded operator preserving the Krein-inner-product possessing the bounded inverse which likewise preserves the Krein-inner-product. Krein isometry in general may be unbounded, but the representors of the Krein-isometric representations induced by Krein-unitary representations are closable on a common dense domain and are invertible on the common domain with the inversions being likewise closable and isometric for the Krein-inner-product, compare [34].

relations for the energy of single photon states obtained by prof. Białynicki (compare [3] and references therein) in showing their compatibility with the causal perturbative QED, [14], [7].

The paper is organized as follows. In Section 2 we define the single photon Krein-isometric representation, i.e. Łopuszański representation, as the representation induced by a Krein-unitary representation of the subgroup equal to the double covering of the euclidean plane, which preserves a zero vector (construction is based on the general theory presented in [34]) and concentrated on the light cone in momentum space. Although the representation preserves the Krein-inner product it is sufficiently singular, being for example unbounded (each boost is represented by a closable unbounded operator), to require a good deal of care. For this reason we have presented all technical details separately in [34]. In the same Section we construct a Krein-isometric representation which is Krein-equivalent (equivalence is given by a Krein isometry preserving core dense domains of the representations which is invertible on the domains) with the Łopuszański representation and has the property that after (inverse) Fourier transforming the single photon states in the momentum picture to the position picture we obtain local transformation law (in the specific case of the Łopuszański representation it is just the four vector transformation law). In doing this we generalize the known construction of representations with local transformation law using our general theory [34], which subsumes the construction for arbitrary spins and ordinary unitary representations as recapitulated e.g. in [17], I.3.3 pp. 31-32 as well as in [1], and includes in particular Krein-isometric representations of (the double cover of) the Poincaré group concentrated on single orbits in the momentum space. The general construction of the local single particle states using the results of [34] is shortly recapitulated in the Appendix for the reader's convenience. In Section 3 we present the construction of the closed physical subspace \mathcal{H}_{tr} of transversal photons together with the unitary representation acting on \mathcal{H}_{tr} which is naturally generated by the action, modulo unphysical states, of the single photon Krein-isometric representation, i. e. by the action of the Łopuszański representation.

2 Łopuszański representation as the single photon representation

Theory of induced representations presented in [34] is effective in treatment of a class of Krein-isometric representations of semi-direct locally compact products $G_1 \ltimes G_2$ of locally compact groups G_1 and G_2 with G_1 being abelian, acting in a Krein space $(\mathcal{H}, \mathfrak{J})$. Krein space is nothing more but an ordinary Hilbert space \mathcal{H} endowed with the fundamental symmetry $\mathfrak{J}^2 = I$, $\mathfrak{J}^* = \mathfrak{J}$, and Krein isometric representation preserves the Krein-inner-product $(\cdot, \mathfrak{J}\cdot)$, but for detailed definition compare Sect. 2 of [34] as the peculiarities like unboundedness (with respect to the ordinary Hilbert space product) cannot be excluded from the outset here in contrast to the ordinary unitary representations. The class

includes Krein-isometric representations of the double covering $T_4 \otimes SL(2, \mathbb{C})$ of the Poincaré group which we need in construction of the free electromagnetic field (and the other zero mass gauge fields of the standard model). We would like to avoid technicalities here but it is important to give some general feelings as to the additional analytic assumptions which allow us to work effectively with the special class of Krein-isometric representations which are of interest in QFT. It should be stressed that the general theory of Krein-isometric representations is substantially richer and much more singular than the theory of ordinary unitary representations. In fact no general theory of such representations has been constructed even for special classical groups which behave regularly (say of Type I). Neumark constructed a series of Krein-unitary representations of the Lorentz group [23] and proved some general theorems on Krein-unitary representations² [24], [25], [26], [27], but only for the case where the fundamental symmetry $\mathfrak{J} = P_+ - P_-$ has P_+ (or P_-) with finite dimensional range³ which unfortunately is useless in QFT. It is therefore important to understand the circumstances (supported by physics) which allows us to treat the case in which both P_+ and P_- have infinite range.

The first circumstance is that within the class of Krein-isometric representations which are of interest we may assume that the representation restricted to the abelian normal subgroup G_1 (which we may identify with translations, let us denote the restriction by T) commutes with the fundamental symmetry \mathfrak{J} (recall the Gupta-Bleuler formalism which at the naive but suggestive level gives momentum operator commuting with the Gupta-Bleuler operator η):

$$\mathfrak{J}T(a) = T(a)\mathfrak{J}, \quad a \in G_1. \quad (1)$$

This circumstance is very important and in particular means that the restriction $T(a)$, $a \in G_1$, of the Krein-isometric representation of $G_1 \otimes G_2$ to the abelian normal factor G_1 (i.e. to the translations) is not only Krein-isometric but also preserves the ordinary Hilbert space inner-product of \mathcal{H} ; or in other words the Krein-isometric representation of $G_1 \otimes G_2$, restricted to abelian normal factor G_1 , gives an ordinary unitary representation of G_1 . This opens us to the full power of the theory of duality for locally compact abelian groups, in particular we may apply the Neumark's theorem about the bi-unique correspondence between unitary representations of locally compact abelian groups G_1 and spectral measures E on their Pontriagin duals \widehat{G}_1 (or when we think about the translation group $G_1 = T_4$ we may identify representations of T_4 with spectral measures in momentum space, via the Fourier transform realizing the duality). This spectral measure E gives the corresponding direct integral decomposition (if one think of $G_1 = T_4$ being the normal factor of $T_4 \otimes SL(2, \mathbb{C})$) of the Hilbert space \mathcal{H} :

$$\mathcal{H} = \int_{sp(P^0, \dots, P^3)} \mathcal{H}_p \, d\mu(p). \quad (2)$$

²Krein-unitary but not unbounded Krein-isometric.

³The case in which the Krein space degenerates to the so called Pontriagin space.

The second circumstance is that we may restrict attention to such Krein-isometric representations of semi-direct products $G_1 \otimes G_2$ for which the restriction $U(\alpha)$, $\alpha \in G_2$ to the second factor G_2 (we may identify G_2 with $SL(2, \mathbb{C})$) is locally bounded with respect to the above mentioned spectral measure E , determined by the restriction $T(a)$, $a \in G_1$, of the representation of $G_1 \otimes G_2$ to the abelian normal factor G_1 , giving the decomposition (2). More precisely: let $\|\cdot\|$ be the ordinary Hilbert space \mathcal{H} norm, then for every compact subset Δ of the dual $\widehat{G_1}$ and every $\alpha \in G_2$ there exists a positive constant $c_{\Delta, \alpha}$ (possibly depending on Δ and α) such that

$$\|U(\alpha)f\| < c_{\Delta, \alpha}\|f\|, \quad (3)$$

for all $f \in \mathcal{H}$ whose spectral support (in the spectral decomposition (2) corresponding to E) is contained within the compact set Δ . This means that all states with bounded momentum whose ordinary Hilbert space norm have common bound are transformed by each $U(\alpha)$ into states whose ordinary norm have common bound. This property is shared by the Lopuszański representation, but its justification is not so easily visible. In our opinion the only reasonable argument for its justification lies in the proof that indeed the application of the Segal second quantization functor Γ to the Lopuszański representation gives the free quantum electromagnetic field, the full proof of which we publish as a separate paper as it is of much more mathematical character, and this is the argument to which we adhere in this paper.

Under the circumstances (1) and (3) Mackey's theory of induced representations may be generalized in treating Krein-isometric representations of semi-direct products $G_1 \otimes G_2$. In particular if we assume in addition that the semi-direct product is regular – for example $T_4 \otimes SL(2, \mathbb{C})$ meets this requirement – in the sense of Mackey (which means that every ergodic measure on $\widehat{G_1}$ with respect to the natural action of G_2 is concentrated on some single orbit in $\widehat{G_1}$ with respect to this action), then we may construct tensor product of such representations and compute effectively their direct integral decompositions into generalized induced representations concentrated on single orbits.

In particular any Krein-isometric representation of $T_4 \otimes SL(2, \mathbb{C})$ which meets the above stated requirements (1) and (3) may be decomposed into direct integral of representations such that for each of them the spectral measure E is concentrated on a single orbit in $\widehat{T_4}$ – which we identify via the Fourier transform with the momentum space – which is ergodic and invariant under the natural action of $SL(2, \mathbb{C})$ on $\widehat{T_4}$. Let $\bar{p} \in \widehat{T_4}$ be any fixed point on the orbit $\mathcal{O}_{\bar{p}}$. Then, regarding its topology and Borel (or Baire) structure, $\mathcal{O}_{\bar{p}}$ is naturally isomorphic to the quotient group $SL(2, \mathbb{C})/G_{\bar{p}}$, where $G_{\bar{p}}$ is the subgroup of $SL(2, \mathbb{C})$ of those elements whose action on $\bar{p} \in \widehat{T_4}$ gives again \bar{p} , i.e. subgroup stationary for \bar{p} . Let $\chi_{\bar{p}}$ be the character in $\widehat{T_4}$ determined by the momentum \bar{p} . Form the subgroup $T_4 \cdot G_{\bar{p}}$ of $T_4 \otimes SL(2, \mathbb{C})$ consisting of all products $a \cdot \gamma$ with $a \in T_4$ and $\gamma \in G_{\bar{p}}$. For each Krein-unitary representation L of the subgroup $G_{\bar{p}}$ the correspondence $a \cdot \gamma \mapsto \chi_{\bar{p}}(a)L(\gamma) = {}_{\chi_{\bar{p}}}L(a \cdot \gamma)$, defines a Krein-unitary representation of $T_4 \cdot G_{\bar{p}}$. It turns out that any Krein-isometric representation

of $T_4 \otimes SL(2, \mathbb{C})$ preserving (1) and (3) which is concentrated on single orbit determined by a closed subgroup $G_{\bar{p}}$ has the form (compare Appendix)

$$\begin{aligned} U(\alpha)\tilde{\psi}(p) &= Q(\gamma(\alpha, p), \bar{p})\tilde{\psi}(\Lambda(\alpha)p), \\ T(a)\tilde{\psi}(p) &= e^{ia \cdot p}\tilde{\psi}(p), \end{aligned} \tag{4}$$

where $p \mapsto \tilde{\psi}(p)$ are the decomposition functions of the elements $\tilde{\psi} \in \mathcal{H}$ with respect to the decomposition (2) determined by the spectral measure E uniquely corresponding to the representation T , with the integral (2) concentrated on the orbit $\mathcal{O}_{\bar{p}}$ and with the measure $d\mu(p)$ equal to the essentially unique measure $d\mu|_{\mathcal{O}_{\bar{p}}}(p)$ on $\mathcal{O}_{\bar{p}}$ invariant with respect to the natural action of $SL(2, \mathbb{C})$; and where $\gamma \mapsto Q(\gamma, \bar{p})$ is a Krein-unitary representation of the subgroup $G_{\bar{p}}$ and $p \mapsto \gamma(\alpha, p) \in G_{\bar{p}}$ a smooth function on the orbit with the values in $G_{\bar{p}}$ (i.e. $Q(\gamma, \bar{p})$ being a Krein unitary operator in the Krein space of the Krein-isometric representation of $G_{\bar{p}}$). Note that in general Krein isometric representation of $T_4 \otimes SL(2, \mathbb{C})$ concentrated on single orbit with the properties (1) and (3) has the property that its restriction T to the abelian normal factor has uniform (generally) infinite multiplicity (compare [34]).

It turns out (Sect. 5 of [34]) that the Krein-isometric representation (4) of $T_4 \otimes SL(2, \mathbb{C})$ is Krein-unitary and unitary equivalent to the Krein-isometric representation $U^{\times_{\bar{p}} L}$ of $T_4 \otimes SL(2, \mathbb{C})$ induced from the Krein-unitary representation ${}_{\times_{\bar{p}}} L$ of the subgroup $T_4 \cdot G_{\bar{p}}$ in the sense defined in Sect. 2 of [34] with $L(\cdot) = \tilde{Q}(\cdot, \bar{p})$.

All these facts about the Krein-isometric representations of the indicated class are very exceptional among Krein-isometric representations, in being closed with respect to the operations of tensoring and of course direct summation, and especially their decomposability properties are very exceptional. Recall in particular that in general the existence of an (even closed) invariant subspace in the Krein space does not guarantee decomposability of the Krein-isometric representation and neither existence of the Krein-self-adjoint and bounded idempotent defining the invariant subspace is guaranteed.

In this paper we are interesting with a specific Krein-isometric representation of $T_4 \otimes SL(2, \mathbb{C})$ induced by a finite dimensional Krein-unitary representation L of the subgroup $G_{(1,0,0,1)}$ stationary for the null vector $\bar{p} = (1, 0, 0, 1)$. In other words the representation T has finite uniform multiplicity, so that the decomposition components (in the sense of von Neumann) may just be regarded as ordinary functions with values in a finite dimensional Hilbert space $\mathcal{H}_{\bar{p}}$ which together with a fundamental symmetry $\mathfrak{J}_{\bar{p}}$ composes the finite dimensional Krein space of the Krein-unitary representation L of the subgroup $G_{(1,0,0,1)}$. We may therefore identify the Hilbert space \mathcal{H} of the representation just with square integrable finite component vector functions with respect to the invariant measure

$$d\mu|_{\mathcal{O}_{\bar{p}}}(\vec{p}) = \frac{d^3 \vec{p}}{2p^0|_{\mathcal{O}_{\bar{p}}}(\vec{p})} = \frac{d^3 \vec{p}}{2\sqrt{m^2 + \vec{p} \cdot \vec{p}}}.$$

which in case of the null vector $\bar{p} = (1, 0, 0, 1)$, i.e. with $\mathcal{O}_{\bar{p}}$ being just the null

cone, is equal

$$d\mu|_{\mathcal{O}_{\bar{p}}}(\vec{p}) = \frac{d^3\vec{p}}{2|\vec{p}|}.$$

The disadvantageous property of the general Krein-isometric (and the same concerns ordinary unitary representation, when $\mathfrak{J} = I$) representation concentrated on a single orbit is that the multiplier in the formula (4) depends non-trivially on the momentum $p \in \mathcal{O}_{\bar{p}}$. This means that in the position picture, i.e. after the application of the Fourier transform, we obtain non local transformation formula, so that the representation after the application of the second quantization functor Γ gives a quantum field with a non local transformation law.

The task of constructing representation which in momentum space have a multiplier in the formula (4) which does not depend on p and give after Fourier transforming a local transformation law has been undertaken for ordinary unitary representations by several authors, compare [17], I.3.3 or [1]. Using our previous results [34] we construct for a wide class of Krein-isometric representations of $T_4 \otimes SL(2, \mathbb{C})$ fulfilling (1) and (3) a Krein-unitary and unitary operator W which transforms the initial Krein-isometric representation (4) (T, U) into Krein-unitary and unitary-equivalent representation $(WTW^{-1}, WUW^{-1}) = (T, WUW^{-1})$ of the form

$$\begin{aligned} WU(\alpha)W^{-1}\tilde{\varphi}(p) &= V(\alpha)\tilde{\varphi}(\Lambda(\alpha)p), \\ WT(a)W^{-1}\tilde{\varphi}(p) &= e^{ia \cdot p}\tilde{\varphi}(p), \end{aligned}$$

for which the multiplier in the formula (4) is an operator (matrix) independent of p and thus after the application of the Fourier transform we obtain local transformation formula. This construction is in particular possible for the Łopuszński representation and uses a Krein-unitary representation V of $SL(2, \mathbb{C})$ acting in the space of the representation $\gamma \mapsto Q(\gamma, \bar{p})$ in (4) of the subgroup $G_{\bar{p}}$ and extends the representation $\gamma \mapsto Q(\gamma, \bar{p})$ to the whole $SL(2, \mathbb{C})$ group.

Construction of this representation and its equivalent version giving local transformation in position picture may be treated as an example of the general construction of local wave function presented in the Appendix based on [34], which extends the results of [1] and [17], I.3.3, to induced Krein-isometric representations in Krein spaces.

Consider the orbit $\mathcal{O}_{(1,0,0,1)}$ of $\bar{p} = (1, 0, 0, 1)$, i.e. positive energy surface of the cone (without the apex $(0, 0, 0, 0)$). The subgroup $G_{(1,0,0,1)} \subset SL(2, \mathbb{C})$ of matrices

$$\gamma = (z, \phi) = \begin{pmatrix} e^{i\phi/2} & e^{i\phi/2}z \\ 0 & e^{-i\phi/2} \end{pmatrix}, \quad 0 \leq \phi < 4\pi, \quad z \in \mathbb{C}$$

is stationary for $(1, 0, 0, 1)$ and is isomorphic to the double covering group⁴ \widetilde{E}_2 of the Euclidean group E_2 of the Euclidean plane.

⁴Equal to the semi-direct product $T_2 \otimes \widetilde{\mathbb{S}^1}$ of the two dimensional translation group T_2 and the double covering of the circle group \mathbb{S}^1 .

As is well known there are no irreducible unitary representations of $G_{(1,0,0,1)}$ besides the infinite dimensional, induced by the characters of the abelian normal subgroup T_2 of $G_{(1,0,0,1)}$ (numbered by a positive real number), and the one dimensional induced by the characters of the abelian subgroup \widetilde{S}^1 and obtained by lifting to $G_{(1,0,0,1)}$ the one dimensional character representations of $G_{(1,0,0,1)}/T_2 \cong \widetilde{S}^1$. And no standard combinations performed on them (direct summation, tensoring, conjugation) can produce after a natural extension V of the resulting representation to the whole $SL(2, \mathbb{C})$ the representation giving the ordinary transformation of a real four vector in Minkowski space (after the natural homomorphic map connecting $SL(2, \mathbb{C})$ to the homogeneous Lorentz group). This is to be expected as the transformation of the four vector under the Lorentz group is not unitary. The situation is different when passing to Krein-unitary representations of $G_{(1,0,0,1)}$.

Namely consider the following representation L of $G_{(1,0,0,1)}$

$$L_\gamma = S(\gamma \otimes \bar{\gamma})S^{-1}, \gamma \in G_{(1,0,0,1)},$$

in \mathbb{C}^4 , where

$$S = \begin{pmatrix} \sqrt{2} & 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & i\sqrt{2} & -i\sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 & -\sqrt{2} \end{pmatrix}$$

is unitary in \mathbb{C}^4 , and where $\bar{\gamma}$ means the ordinary complex conjugation: if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } \bar{\gamma} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}.$$

If we introduce to \mathbb{C}^4 the ordinary inner product and the following fundamental symmetry operator

$$\mathfrak{J}_{\bar{p}} = \mathfrak{J}_{(1,0,0,1)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5)$$

then the representation L of $G_{(1,0,0,1)} = G_{\bar{p}}$ becomes Krein-unitary in the Krein space $(\mathbb{C}^4, \mathfrak{J}_{\bar{p}})$:

$$L \mathfrak{J}_{\bar{p}} L^* \mathfrak{J}_{\bar{p}} = \mathbf{1}_4, \text{ and } \mathfrak{J}_{\bar{p}} L^* \mathfrak{J}_{\bar{p}} L = \mathbf{1}_4,$$

where L_γ^* denotes the ordinary adjoint operator of L_γ with respect to the ordinary inner product in \mathbb{C}^4 .

The function $p \mapsto \beta(p) \in SL(2, \mathbb{C})$, fulfilling $\beta(p)^{-1} \widehat{p} (\beta(p)^{-1})^* = \widehat{p}$ on the orbit $\mathcal{O}_{(1,0,0,1)}$, where $\widehat{p} = (p^0, \dots, p^3) = p^0 \sigma_0 + p^1 \sigma_1 + p^2 \sigma_2 + p^3 \sigma_3$ with Pauli matrices σ_μ , may be chosen to be equal

$$\beta(p) = \begin{pmatrix} r^{-1/2} \cos \frac{\theta}{2} e^{-i\frac{\varphi}{2}} & -ir^{-1/2} \sin \frac{\theta}{2} e^{i\frac{\varphi}{2}} \\ -ir^{1/2} \sin \frac{\theta}{2} e^{-i\frac{\varphi}{2}} & r^{1/2} \cos \frac{\theta}{2} e^{i\frac{\varphi}{2}} \end{pmatrix},$$

where

$$p = \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} = \begin{pmatrix} r \\ r \sin \theta \sin \vartheta \\ r \sin \theta \cos \vartheta \\ r \cos \theta \end{pmatrix} \in \mathcal{O}_{(1,0,0,1)}, \quad 0 \leq \theta < \pi, 0 \leq \vartheta < 2\pi, r > 0.$$

Now we construct, like in the Appendix, the Krein-isometric representation of $T_4 \otimes SL(2, \mathbb{C})$ induced by the the Krein-unitary representation \mathbf{L} , putting there \mathbf{L}_γ for $Q(\gamma, \bar{p})$ with $\bar{p} = (1, 0, 0, 1)$. Let us denote the representation by $U^{(1,0,0,1)} \mathbf{L}$ and call the *Lopuszański representation*. By Section 5 of [34] (and the Appendix), it is Krein-unitary equivalent to the Krein-isometric representation of $T^4 \otimes SL(2, \mathbb{C})$ induced⁵ by the representation ${}_{(1,0,0,1)} \mathbf{L} = \chi_{\bar{p}} \mathbf{L}$:

$$a \cdot \gamma \mapsto \chi_{\bar{p}}(a) \mathbf{L}_\gamma,$$

of the subgroup $T_4 \cdot G_{\bar{p}} \subset T_4 \otimes SL(2, \mathbb{C})$.

Now we define the following extension

$$V(\alpha) = S(\alpha \otimes \bar{\alpha}) S^{-1}, \alpha \in SL(2, \mathbb{C}),$$

of the representation \mathbf{L} , to the whole $SL(2, \mathbb{C})$ group, which is likewise Krein-unitary in $(\mathbb{C}^4, \mathfrak{J}_{\bar{p}})$:

$$V(\alpha) \mathfrak{J}_{\bar{p}} V(\alpha)^* \mathfrak{J}_{\bar{p}} = \mathbf{1}_4, \quad \text{and} \quad \mathfrak{J}_{\bar{p}} V(\alpha)^* \mathfrak{J}_{\bar{p}} V(\alpha) = \mathbf{1}_4, \quad \alpha \in SL(2, \mathbb{C}).$$

Moreover $\alpha \mapsto V(\alpha)$ gives a natural homomorphism of the $SL(2, \mathbb{C})$ onto the proper orthochronous Lorentz group in the Minkowski vector space, i.e. each $V(\alpha)$, $\alpha \in SL(2, \mathbb{C})$, is a real Lorentz transformation. It is customary to write $V(\alpha)$ as the corresponding Lorentz transformation $\Lambda(\alpha)$. Because we have already occupied the notation $\Lambda(\alpha)$ for a natural antihomomorphism Λ , we have $V(\alpha) = \Lambda(\alpha^{-1})$ in our notation.

With the extension V at our disposal, we apply to the elements $\tilde{\psi}$ of the space of the Łopuszański representation $U^{(1,0,0,1)} \mathbf{L}$ the Krein unitary and unitary transformation $W : \tilde{\psi} \mapsto \tilde{\varphi}$ presented in the Appendix, having the property that the Fourier transform (33) φ has the local transformation law. Namely the representation $W U^{(1,0,0,1)} \mathbf{L} W^{-1}$ acts as follows

$$\begin{aligned} W U_{0,\alpha}^{(1,0,0,1)} \mathbf{L} W^{-1} \tilde{\varphi}(p) &= U(\alpha) \tilde{\varphi}(p) = V(\alpha) \tilde{\varphi}(\Lambda(\alpha)p), \\ W U_{a,1}^{(1,0,0,1)} \mathbf{L} W^{-1} \tilde{\varphi}(p) &= T(a) \tilde{\varphi}(p) = e^{ia \cdot p} \tilde{\varphi}(p). \end{aligned} \tag{6}$$

Therefore the Fourier transform (33) φ of $\tilde{\varphi} = W \tilde{\psi}$ has the the following local transformation law

$$U(\alpha) \varphi(x) = V(\alpha) \varphi(x \Lambda(\alpha^{-1})) = \Lambda(\alpha^{-1}) \varphi(x \Lambda(\alpha^{-1})), \quad T(a) \varphi(x) = \varphi(x - a).$$

⁵In the sense of definition placed in Section 2 of [34], which is a generalization of the Mackey's induced representation.

of a four vector field on the Minkowski manifold. Because by construction $\tilde{\varphi}$ are concentrated on the orbit $\mathcal{O}_{(1,0,0,1)}$, it follows that the elements $\varphi \in \mathcal{H}''$ are the positive energy (distributional) solutions of the ordinary wave equation with zero mass

$$\partial^\mu \partial_\mu \varphi = 0.$$

The explicit form of the Krein space structure $(\mathcal{H}', \mathfrak{J}')$ of the representation space of the representation $WU^{(1,0,0,1)} \bar{L} W^{-1}$ can be obtained by substitution of the explicit formulas for the function $p \mapsto \beta(p)$ and the extension V into the formulas written in the Appendix.

In particular the inner product of $\tilde{\varphi} = W\tilde{\psi}$ and $\tilde{\varphi}' = W\tilde{\psi}'$ is equal

$$\begin{aligned} (\tilde{\varphi}, \tilde{\varphi}') &= \int_{sp(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \left(\tilde{\varphi}(p), \tilde{\varphi}'(p) \right)_p d\mu|_{\mathcal{O}_{\bar{p}}}(p) \\ &= \int_{sp(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \left(\tilde{\varphi}(p), V(\beta(p))^* V(\beta(p)) \tilde{\varphi}'(p) \right)_{\mathcal{H}_{\bar{p}}} d\mu|_{\mathcal{O}_{\bar{p}}}(p) \\ &= \int_{\mathcal{O}_{\bar{p}}} \left(\tilde{\varphi}(p), B(p) \tilde{\varphi}'(p) \right)_{\mathbb{C}^4} d\mu|_{\mathcal{O}_{\bar{p}}}(p), \\ &= \int_{\mathbb{R}^3} \left(\tilde{\varphi}(\vec{p}, p^0(\vec{p})), (B\tilde{\varphi}')(\vec{p}, p^0(\vec{p})) \right)_{\mathbb{C}^4} d^3 p = (\tilde{\varphi}, B\tilde{\varphi}')_{\oplus L^2(\mathbb{R}^3)}, \\ &\quad d\mu|_{\mathcal{O}_{\bar{p}}}(\vec{p}) = \frac{d^3 p}{2p^0(\vec{p})}, p^0(\vec{p}) = (\vec{p} \cdot \vec{p})^{1/2}, \quad (7) \end{aligned}$$

where we have introduced the matrix

$$B(p) = V(\beta(p))^* V(\beta(p))$$

depending on $p \in \mathcal{O}_{\bar{p}}$, strictly positive (invertible) on $\mathcal{O}_{\bar{p}}$ and the operator B of point-wise multiplication by the matrix

$$\frac{1}{2p^0(\vec{p})} B(\vec{p}, p^0(\vec{p})), \quad (8)$$

on the Hilbert space $\oplus L^2(\mathbb{R}^3) = L^2(\mathbb{R}^3; \mathbb{C}^4)$ with respect to the ordinary invariant Lebesgue measure $d^3 \mathbf{p}$ on \mathbb{R}^3 (the direct sum \oplus is over the four components of the function $\tilde{\varphi}$), in order to simplify notation of the formulas which are to follow in the remaining part of this paper.

The fundamental symmetry operator \mathfrak{J}' is given by the point-wise multiplication by the following operator

$$\mathfrak{J}'_p = V(\beta(p))^{-1} \mathfrak{J}_{\bar{p}} V(\beta(p)).$$

Because for each $p \in \mathcal{O}_{\bar{p}}$ the matrix operator $V(\beta(p))$ (and the same of course holds for $V(\beta(p))^*$) is by construction Krein-unitary in the Krein space $(\mathbb{C}^4, \mathfrak{J}_{\bar{p}}) =$

$(\mathcal{H}_{\bar{p}}, \mathfrak{J}_{\bar{p}})$ of the representation \mathbf{L} , then the Krein product in $(\mathcal{H}', \mathfrak{J}')$ is given by the following formula

$$\begin{aligned}
(\tilde{\varphi}, \mathfrak{J}' \tilde{\varphi}') &= \int_{sp(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \left(\tilde{\varphi}(p), V(\beta(p))^* V(\beta(p)) \mathfrak{J}'_p \tilde{\varphi}'(p) \right)_{\mathcal{H}_{\bar{p}}} d\mu|_{\mathcal{O}_{\bar{p}}}(p) \\
&= \int_{\mathcal{O}_{\bar{p}}} \left(\tilde{\varphi}(p), V(\beta(p))^* V(\beta(p)) V(\beta(p))^{-1} \mathfrak{J}_{\bar{p}} V(\beta(p)) \tilde{\varphi}'(p) \right)_{\mathbb{C}^4} d\mu|_{\mathcal{O}_{\bar{p}}}(p) \\
&= \int_{\mathcal{O}_{\bar{p}}} \left(\tilde{\varphi}(p), \mathfrak{J}_{\bar{p}} \tilde{\varphi}'(p) \right)_{\mathbb{C}^4} d\mu|_{\mathcal{O}_{\bar{p}}}(p),
\end{aligned} \tag{9}$$

because $V(\beta(p))^* \mathfrak{J}_{\bar{p}} V(\beta(p)) = \mathfrak{J}_{\bar{p}}$.

Introducing the coordinates \vec{p} on $\mathcal{O}_{\bar{p}}$ and regarding any function $p \mapsto \tilde{\varphi}(p)$ on $\mathcal{O}_{\bar{p}}$ as a function $\vec{p} \mapsto \tilde{\varphi}(\vec{p}) = \tilde{\varphi}(\vec{p}, p^0(\vec{p}))$ with $p^0(\vec{p})$ as in (7), the last formula (9) may be written as

$$(\tilde{\varphi}, \mathfrak{J}' \tilde{\varphi}') = (\tilde{\varphi}, B \mathfrak{J}' \tilde{\varphi}')_{\oplus L^2(\mathbb{R}^3)} = (\sqrt{B} \tilde{\varphi}, \sqrt{B} \mathfrak{J}' \tilde{\varphi}')_{\oplus L^2(\mathbb{R}^3)} = (\tilde{\varphi}, \mathfrak{J}_{\bar{p}} \tilde{\varphi}')_{\oplus L^2(\mathbb{R}^3, d\mu|_{\mathcal{O}_{\bar{p}}})}, \tag{10}$$

where the last inner product

$$(\cdot, \cdot)_{\oplus L^2(\mathbb{R}^3, d\mu|_{\mathcal{O}_{\bar{p}}})}$$

is with respect to the measure

$$d\mu = \frac{d^3 p}{2p^0(\vec{p})}, \quad p^0(\vec{p}) = (\vec{p} \cdot \vec{p})^{1/2},$$

on \mathbb{R}^3 , and where B is the positive self-adjoint operator on $\oplus L^2(\mathbb{R}^3)$ introduced above and \sqrt{B} is its square root equal to the operator of point-wise multiplication by the matrix

$$\frac{1}{\sqrt{2p^0(\vec{p})}} \sqrt{B(\vec{p}, p^0(\vec{p}))},$$

with $\sqrt{B(\vec{p}, p^0(\vec{p}))}$ being the square root of the positive matrix $B(\vec{p}, p^0(\vec{p}))$.

Krein-isometric and Krein-unitary representations in a Krein space $(\mathcal{H}, \mathfrak{J})$ allows the specific kind of conjugation, which is trivial for ordinary unitary representations when $\mathfrak{J} = \mathbf{1}_{\mathcal{H}}$. Namely for every representation U of this kind in the Krein space $(\mathcal{H}, \mathfrak{J})$, the ordinary Hilbert space adjoint operation $*$ and passing to the inverse, i.e. $U^{*-1} = \mathfrak{J} U \mathfrak{J}$, is well defined, which is nontrivial for Krein-isometric representation, compare [34], Sect. 2. Moreover $U^{*-1} = \mathfrak{J} U \mathfrak{J}$ defines another Krein-isometric (resp. Krein unitary) representation with respect to the same Krein structure, compare [34], Sect. 2, which is unitary and

Krein-unitary equivalent to the initial representation U , with the equivalence given by the fundamental symmetry \mathfrak{J} itself, and \mathfrak{J} is by construction unitary and Krein-unitary.

In particular together with the Krein-isometric representation $WU^{(1,0,0,1)}\mathbb{L}W^{-1}$ in the Krein space $(\mathcal{H}', \mathfrak{J}')$ just constructed, there acts in the same Krein space $(\mathcal{H}', \mathfrak{J}')$ the naturally conjugate Krein isometric representation

$$[WU^{(1,0,0,1)}\mathbb{L}W^{-1}]^{*-1} = \mathfrak{J}'WU^{(1,0,0,1)}\mathbb{L}W^{-1}\mathfrak{J}'$$

unitary and Krein-unitary equivalent to $WU^{(1,0,0,1)}\mathbb{L}W^{-1}$, with the equivalence given by the fundamental symmetry \mathfrak{J}' itself. Because we have explicitly computed \mathfrak{J}' and $WU^{(1,0,0,1)}\mathbb{L}W^{-1}$ we also know the explicit formula for the action of $[WU^{(1,0,0,1)}\mathbb{L}W^{-1}]^{*-1}$. Namely we have

$$\begin{aligned} [WU^{(1,0,0,1)}\mathbb{L}W^{-1}]^{*-1}\tilde{\varphi}(p) &= (\mathfrak{J}'WU^{(1,0,0,1)}\mathbb{L}W^{-1}\mathfrak{J}')\tilde{\varphi}(p) \\ &= V(\beta(p))^{-1}V(\beta(p))^{*-1}V(\alpha)^{*-1}V(\beta(\Lambda(\alpha)p))^*V(\beta(\Lambda(\alpha)p))\tilde{\varphi}(\Lambda(\alpha)p). \end{aligned}$$

Before passing to quantization, we give here several formulas which will be useful in further computations. First let us note the simple formula for the Krein inner product in the Krein space $(\mathcal{H}'', \mathfrak{J}'')$ of all Fourier transforms φ , given by (33), of the elements $\tilde{\varphi}$ of the Krein space $(\mathcal{H}', \mathfrak{J}')$ of the representation $WU^{(1,0,0,1)}\mathbb{L}W^{-1}$. Namely easy computation gives

$$\begin{aligned} (\varphi, \mathfrak{J}''\varphi) &= i \int_{t=\text{const.}} \left\{ \overline{\varphi(x)} \partial_t (\mathfrak{J}_{\bar{p}}\varphi')(x) - \overline{\partial_t \varphi(x)} \mathfrak{J}_{\bar{p}}\varphi'(x) \right\} d^3x \\ &= -ig^{\mu\nu} \int_{t=\text{const.}} \left\{ \overline{\varphi_\mu(x)} \partial_t \varphi'_\nu(x) - \overline{\partial_t \varphi_\mu(x)} \varphi'_\nu(x) \right\} d^3x \quad (11) \end{aligned}$$

Next we give explicit formulas for $V(\beta(p))^{-1}$, $B(p) = V(\beta(p))^*V(\beta(p))$ and $\sqrt{B(p)}$, $p \in \mathcal{O}_{(1,0,0,1)}$, and give their useful properties.

$$\begin{aligned} V(\beta(p))^{-1} &= \begin{pmatrix} \frac{r^{-1}+r}{2} & 0 & 0 & -\frac{r^{-1}-r}{2} \\ -\frac{r^{-1}-r}{2} \frac{p^1}{r} & \frac{p^2}{\sqrt{(p^1)^2+(p^2)^2}} & \frac{p^1}{\sqrt{(p^1)^2+(p^2)^2}} \frac{p^3}{r} & \frac{r^{-1}+r}{2} \frac{p^1}{r} \\ -\frac{r^{-1}-r}{2} \frac{p^2}{r} & -\frac{p^1}{\sqrt{(p^1)^2+(p^2)^2}} & \frac{p^2}{\sqrt{(p^1)^2+(p^2)^2}} \frac{p^3}{r} & \frac{r^{-1}+r}{2} \frac{p^2}{r} \\ -\frac{r^{-1}-r}{2} \frac{p^3}{r} & 0 & -\frac{\sqrt{(p^1)^2+(p^2)^2}}{r} & \frac{r^{-1}+r}{2} \frac{p^3}{r} \end{pmatrix} \\ &= \begin{pmatrix} \frac{r^{-1}+r}{2} & 0 & 0 & -\frac{r^{-1}-r}{2} \\ -\frac{r^{-1}-r}{2} \sin \theta \sin \vartheta & \cos \vartheta & \cos \theta \sin \vartheta & \frac{r^{-1}+r}{2} \sin \theta \sin \varphi \\ -\frac{r^{-1}-r}{2} \sin \theta \cos \varphi & -\sin \vartheta & \cos \theta \cos \vartheta & \frac{r^{-1}+r}{2} \sin \theta \cos \vartheta \\ -\frac{r^{-1}-r}{2} \cos \theta & 0 & -\sin \theta & \frac{r^{-1}+r}{2} \cos \theta \end{pmatrix}. \end{aligned}$$

$$B(p) = V(\beta(p))^* V(\beta(p)) =$$

$$\begin{pmatrix} \frac{r^{-2}+r^2}{2} & \frac{r^{-2}-r^2}{2r} p^1 & \frac{r^{-2}-r^2}{2r^2} p^2 & \frac{r^{-2}-r^2}{2r^2} p^3 \\ \frac{r^{-2}-r^2}{2} p^1 & \frac{r^{-2}+r^2-2}{2r^2} p^1 p^1 + 1 & \frac{r^{-2}+r^2-2}{2r^2} p^1 p^2 & \frac{r^{-2}+r^2-2}{2r^2} p^1 p^3 \\ \frac{r^{-2}-r^2}{2r} p^2 & \frac{r^{-2}+r^2-2}{2r^2} p^2 p^1 & \frac{r^{-2}+r^2-2}{2r^2} p^2 p^2 + 1 & \frac{r^{-2}+r^2-2}{2r^2} p^2 p^3 \\ \frac{r^{-2}-r^2}{2r} p^3 & \frac{r^{-2}+r^2-2}{2r^2} p^3 p^1 & \frac{r^{-2}+r^2-2}{2r^2} p^3 p^2 & \frac{r^{-2}+r^2-2}{2r^2} p^3 p^3 + 1 \end{pmatrix} \quad (12)$$

$$= \begin{pmatrix} \frac{r^{-2}+r^2}{2} & \frac{r^{-2}-r^2}{2} \sin \theta \sin \vartheta & \frac{r^{-2}-r^2}{2} \sin^2 \theta \sin^2 \vartheta + \cos^2 \theta \sin^2 \vartheta + \cos^2 \theta & \dots \\ \frac{r^{-2}-r^2}{2} \sin \theta \sin \vartheta & \frac{r^{-2}+r^2}{2} \sin^2 \theta \cos^2 \vartheta \sin \vartheta + \cos^2 \theta \sin \vartheta \cos \vartheta - \sin \vartheta \cos \vartheta & \dots & \dots \\ \frac{r^{-2}-r^2}{2} \sin \theta \cos \vartheta & \frac{r^{-2}+r^2}{2} \sin^2 \theta \cos \vartheta \sin \vartheta + \cos^2 \theta \sin \vartheta \cos \vartheta - \sin \vartheta \cos \vartheta & \dots & \dots \\ \frac{r^{-2}-r^2}{2} \cos \theta & \frac{r^{-2}+r^2}{2} \sin \theta \cos \theta \sin \vartheta - \sin \theta \cos \theta \sin \vartheta & \dots & \dots \end{pmatrix}$$

$$\dots \begin{pmatrix} \frac{r^{-2}-r^2}{2} \sin \theta \cos \vartheta & \frac{r^{-2}-r^2}{2} \cos \theta \\ \frac{r^{-2}+r^2}{2} \sin^2 \theta \cos \vartheta \sin \vartheta + \cos^2 \theta \sin \vartheta \cos \vartheta - \sin \vartheta \cos \vartheta & \frac{r^{-2}+r^2}{2} \sin \theta \cos \theta \sin \vartheta - \sin \theta \cos \theta \sin \vartheta \\ \frac{r^{-2}+r^2}{2} \sin^2 \theta \cos^2 \vartheta + \cos^2 \theta \cos^2 \vartheta + \sin^2 \vartheta & \frac{r^{-2}+r^2}{2} \sin \theta \cos \theta \cos \vartheta - \sin \theta \cos \theta \cos \vartheta \\ \frac{r^{-2}+r^2}{2} \sin \theta \cos \theta \cos \vartheta - \sin \theta \cos \theta \cos \vartheta & \frac{r^{-2}+r^2}{2} p^3 \cos^2 \theta + \sin^2 \theta \end{pmatrix}.$$

The orthonormal (with respect to the ordinary inner product in \mathbb{C}^4) system $\{w_\lambda(p)\}$ of eigenvectors of the operator matrix $B(p) = V(\beta(p))^* V(\beta(p))$ in \mathbb{C}^4 , corresponding to the eigenvalues $\lambda(p) \in \{1, 1, r^{-2}, r^2\}$ has the form

$$w_1^+(p) = \begin{pmatrix} 0 \\ \frac{p^2}{\sqrt{(p^1)^2 + (p^2)^2}} \\ -\frac{p^1}{\sqrt{(p^1)^2 + (p^2)^2}} \\ 0 \end{pmatrix}, w_1^-(p) = \begin{pmatrix} 0 \\ \frac{p^1 p^3}{\sqrt{(p^1)^2 + (p^2)^2} r} \\ \frac{p^2 p^3}{\sqrt{(p^1)^2 + (p^2)^2} r} \\ -\frac{\sqrt{(p^1)^2 + (p^2)^2}}{r} \end{pmatrix},$$

$$w_{r^{-2}}(p) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \frac{p^1}{r} \\ \frac{1}{\sqrt{2}} \frac{p^2}{r} \\ \frac{1}{\sqrt{2}} \frac{p^3}{r} \end{pmatrix}, w_{r^2}(p) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \frac{p^1}{r} \\ -\frac{1}{\sqrt{2}} \frac{p^2}{r} \\ -\frac{1}{\sqrt{2}} \frac{p^3}{r} \end{pmatrix}$$

There are two transversal eigenvectors $w_1^+(p), w_1^-(p)$ to the constant eigenvalue 1, both of pure space direction and both orthogonal to the space part $(0, \vec{p})$ of the momentum direction of the corresponding momentum $p = (p^0, \vec{p}) \in \mathcal{O}_{(1,0,0,1)}$. The eigenvector $w_{r^{-2}}(p)$ corresponding to the eigenvalue $r^{-2} = (p^0)^{-2} = (\vec{p} \cdot \vec{p})^{-1}$, has the same direction as the corresponding momentum $p = (p^0, \vec{p}) \in \mathcal{O}_{(1,0,0,1)}$, and $w_{r^2}(p)$ has the same direction as $(p^0, -\vec{p})$, where $p = (p^0, \vec{p}) \in \mathcal{O}_{(1,0,0,1)}$ is the corresponding momentum. Note that the linear combinations $w_{r^{-2}}(p) + w_{r^2}(p)$ and $w_{r^{-2}}(p) - w_{r^2}(p)$ give respectively the purely time-like vector of direction the same as $(p^0, 0)$ and a purely longitudinal vector

of direction the same as $(0, \vec{p})$, where $p = (p^0, \vec{p}) \in \mathcal{O}_{(1,0,0,1)}$ is the corresponding momentum vector.

The square root of $B(p) = V(\beta(p))^* V(\beta(p))$ is equal

$$\begin{aligned} \sqrt{V(\beta(p))^* V(\beta(p))} &= \sqrt{B(p)} \\ &= \begin{pmatrix} \frac{r^{-1}+r}{2} & \frac{r^{-1}-r}{2} \frac{p^1}{r} & \frac{r^{-1}-r}{2} \frac{p^2}{r} & \frac{r^{-1}-r}{2} \frac{p^3}{r} \\ \frac{r^{-1}-r}{2} \frac{p^1}{r} & \frac{r^{-1}+r-2}{2} \frac{p^1}{r} \frac{p^1}{r} + 1 & \frac{r^{-1}+r-2}{2} \frac{p^1}{r} \frac{p^2}{r} & \frac{r^{-1}+r}{2} \frac{p^1}{r} \frac{p^3}{r} \\ \frac{r^{-1}-r}{2} \frac{p^2}{r} & \frac{r^{-1}+r-2}{2} \frac{p^2}{r} \frac{p^1}{r} & \frac{r^{-1}+r-2}{2} \frac{p^2}{r} \frac{p^2}{r} + 1 & \frac{r^{-1}+r}{2} \frac{p^2}{r} \frac{p^3}{r} \\ \frac{r^{-1}-r}{2} \frac{p^3}{r} & \frac{r^{-1}+r-2}{2} \frac{p^3}{r} \frac{p^1}{r} & \frac{r^{-1}+r-2}{2} \frac{p^3}{r} \frac{p^2}{r} & \frac{r^{-1}+r-2}{2} \frac{p^3}{r} \frac{p^3}{r} + 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{r^{-1}+r}{2} & \frac{r^{-1}-r}{2} \sin \theta \sin \vartheta & \frac{r^{-1}-r}{2} \sin \theta \cos \vartheta & \frac{r^{-1}-r}{2} \cos \theta \\ \frac{r^{-1}-r}{2} \sin \theta \sin \vartheta & \frac{r^{-1}+r-2}{2} \sin^2 \theta \sin^2 \vartheta + 1 & \frac{r^{-1}+r-2}{2} \sin^2 \theta \sin \vartheta \cos \vartheta & \frac{r^{-1}+r-2}{2} \sin \theta \cos \theta \sin \vartheta \\ \frac{r^{-1}-r}{2} \sin \theta \cos \vartheta & \frac{r^{-1}+r-2}{2} \sin^2 \theta \cos \vartheta \sin \vartheta & \frac{r^{-1}+r-2}{2} \sin^2 \theta \cos^2 \vartheta + 1 & \frac{r^{-1}+r}{2} \sin \theta \cos \theta \cos \vartheta \\ \frac{r^{-1}-r}{2} \cos \theta & \frac{r^{-1}+r-2}{2} \sin \theta \cos \theta \sin \vartheta & \frac{r^{-1}+r-2}{2} \sin \theta \cos \theta \cos \vartheta & \frac{r^{-1}+r-2}{2} \cos^2 \theta + 1 \end{pmatrix}. \end{aligned} \quad (13)$$

By construction $V(\beta(p))$, $V(\beta(p))^* = V(\beta(p))^T$ and their inverses are at every $p \in \mathcal{O}_{(1,0,0,1)}$ Krein unitary, as matrix operators in $(\mathcal{H}_{\vec{p}}, \mathfrak{J}_{\vec{p}}) = (\mathbb{C}^4, \mathfrak{J}_{\vec{p}})$, i.e they are real Lorentz transformations. It is less trivial, but may be checked directly that for every $p \in \mathcal{O}_{(1,0,0,1)}$ the operator $\sqrt{B(p)}$ is also Krein unitary in $(\mathbb{C}^4, \mathfrak{J}_{\vec{p}})$. Thus we have the formulas

$$\begin{aligned} V(\beta(p)) \mathfrak{J}_{\vec{p}} V(\beta(p))^* \mathfrak{J}_{\vec{p}} &= \mathbf{1}_4, \text{ and } \mathfrak{J}_{\vec{p}} V(\beta(p))^* \mathfrak{J}_{\vec{p}} V(\beta(p)) = \mathbf{1}_4, \text{ and} \\ \sqrt{B(p)} \mathfrak{J}_{\vec{p}} \sqrt{B(p)} \mathfrak{J}_{\vec{p}} &= \mathbf{1}_4, \quad p \in \mathcal{O}_{(1,0,0,1)}. \end{aligned} \quad (14)$$

Although the properties are simple consequences of definitions (possibly with the exception of the last one) they will be of use in further computations.

DEFINITION OF THE KREIN-HILBERT SPACE $(\mathcal{H}', \mathfrak{J}')$ WHICH IS THEN SUBJECT TO THE SECOND QUANTIZATION FUNCTOR Γ

Now to the Hilbert space \mathcal{H}' , or more precisely to the Krein space $(\mathcal{H}', \mathfrak{J}')$ of the representation $WU^{(1,0,0,1)} \mathbf{L} W^{-1}$ and *eo ipso* of the representation

$$[WU^{(1,0,0,1)} \mathbf{L} W^{-1}]^{*-1},$$

we apply the Segal's bosonic second quantization functor Γ . The Krein space $(\mathcal{H}', \mathfrak{J}') = (W\mathcal{H}, W\mathfrak{J}W^{-1})$ of the elements $\tilde{\varphi} = W\psi$ of the representation

$$WU^{(1,0,0,1)} \mathbf{L} W^{-1}$$

may be identified, via the Fourier transform (33) with the Hilbert space \mathcal{H}'' , or more precisely with the Krein space $(\mathcal{H}'', \mathfrak{J}'')$ of positive energy solutions φ of the wave equation

$$g^{\mu\nu} \partial_\mu \partial_\nu \varphi = 0, \quad (15)$$

as a consequence of the fact that $\tilde{\varphi} \in \mathcal{H}'$ are concentrated on the cone $\mathcal{O}_{(1,0,0,1)} = \mathcal{O}_{(1,0,0,1)}$. Although it is well known that the equation (15) only apparently gives a local law for dynamics in terms of a local equation. Indeed because only the positive energy solutions⁶ are admitted the quantities φ and $\partial_t \varphi$ are not independent on a fixed time surface. The differentiation ∂_t in momentum space is equal to the operator of multiplication by $-i\sqrt{\vec{p} \cdot \vec{p}}$, which in position picture at fixed time corresponds to a convolution with the non-local integral kernel⁷

$$K(\vec{x} - \vec{x}') = -i(2\pi)^{-3/2} \int_{\mathbb{R}^3} \sqrt{\vec{p} \cdot \vec{p}} e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} d^3p,$$

exactly as for the spin-less massive particles (compare e. g. [17], I. 3.3.).

Unfortunately the ordinary Hilbert space inner product $(\varphi, \varphi') = (\tilde{\varphi}, \tilde{\varphi}')$, when expressed in terms of φ and φ' , involves unpleasant kernel. This is however not so important as the Hilbert space inner product plays the (important but) only technical role of controlling all the analytical subtleties. It is the Krein inner product $(\varphi, \mathfrak{J}''\varphi') = (\tilde{\varphi}, \mathfrak{J}'\tilde{\varphi}')$ which serves to compute probabilities on the subspace of physical states on which it is positive definite, and it is nice to have the relatively simple and explicit formula (11) for the Krein-inner product in the Krein space $(\mathcal{H}'', \mathfrak{J}'')$ expressed in terms of position wave functions φ, φ' .

We must be careful in preparing the fields as Wightman operator-valued distributions. This can be achieved by application of the Schwartz-Woronowicz kernel theorem to the test function spaces \mathcal{S}_0 and \mathcal{S}_{00} mentioned to above to the operator valued distribution (quantum vector potential)

$$A(\varphi) = A^\mu(\varphi_\mu) = a(\tilde{\varphi}|_{\mathcal{O}_{\vec{p}}}) + \eta a(\tilde{\varphi}|_{\mathcal{O}_{\vec{p}}})^+ \eta,$$

where $a(\varphi), a(\varphi)^+$ are the annihilation-creation families of operators constructed by the application of the Segal second quantization functor Γ to the *conjugated* Łopuszański representation, η is the Gupta-Bleuler operator equal $\Gamma(\mathfrak{J}')$, and where $\varphi \in \mathcal{S}_{00}(\mathbb{R}^4)$, its Fourier transform $\tilde{\varphi}$ belongs to $\mathcal{S}_0(\mathbb{R}^4)$ and where $\tilde{\varphi} \mapsto \tilde{\varphi}|_{\mathcal{O}_{\vec{p}}}$ is the restriction to the cone, which turns out to be indeed a continuous map of nuclear spaces $\mathcal{S}_0(\mathbb{R}^4) \rightarrow \mathcal{S}_0(\mathbb{R}^3)$ (this requires a proof).

It turns out that indeed the commutator

$$[A(\varphi), A(\varphi')]$$

defines the kernel distribution equal to the Pauli-Jordan function multiplied by the minkowskian metric; in the proof one can apply e.g. the Kernel theorem as stated in [36] (the ordinary Schwartz kernel theorem is not sufficient for the construction of the Wick product); and it follows that $A(\varphi)$ is the Wightman field transforming locally as a four vector field. The full construction of the free

⁶In the construction of the positive energy field via the second quantization functor applied to the space $(\mathcal{H}', \mathfrak{J}')$. In the construction of the negative energy field the roles of positive and negative energy is interchanged.

⁷Already the definition of the kernel necessitates a special care, and may be defined in the distributional sense

quantum electromagnetic field (and the quantum vector potential field) through the application of the Segal's functor Γ of second quantization to the conjugate Lopuszański representation we are going to publish in a separate paper.

It should be stressed that already the elements $\tilde{\varphi}$ of the single particle space of the Lopuszański representation (and its conjugation) in the momentum picture do not in general fulfil the condition $p^\mu \tilde{\varphi}_\mu = 0$, so that in general their Fourier transforms φ do not preserve the Lorentz condition $\partial^\mu \varphi_\mu = 0$. This corresponds to the well known fact that the Lorentz condition cannot be preserved as an operator equation. It can be preserved in the sense of the Krein-product average on a subspace of Lorentz states which arise from the closed subspace \mathcal{H}_{tr} of the so called transversal states together with all their images under the action of the Lopuszański representation and its conjugation. We are now going to define the closed subspace \mathcal{H}_{tr} .

3 Construction of the physical subspace \mathcal{H}_{tr} of transversal photon states

Note that the operator B of multiplication by the positive selfadjoint matrix (8) is selfadjoint in the Hilbert space $\oplus L^2(\mathbb{R}^3) = L^2(\mathbb{R}^3, \mathbb{C}^4)$ with respect to the ordinary invariant Lebesgue measure $d^3\mathbf{p}$ on \mathbb{R}^3 (the direct sum \oplus is over the four components of the function $\tilde{\varphi}$), and that the Hilbert space inner product in the single-photon state space \mathcal{H}' is equal $(\cdot, \cdot) = (\cdot, B\cdot)_{L^2(\mathbb{R}^3, \mathbb{C}^4)}$. The unitary operator which has the direct integral decomposition

$$\int_{\mathcal{O}_{\bar{p}}} U_{\mathbf{p}} d^3\mathbf{p} \quad (16)$$

(in the integral we use the spatial momentum coordinates \mathbf{p} on the cone $\mathcal{O}_{\bar{p}}$, and the integral may be treated as an integral on \mathbb{R}^3) with each component $U_{\mathbf{p}}$ being a unitary matrix operator in \mathbb{C}^4 transforming the standard basis in \mathbb{C}^4 into the basis⁸ $w_1^+(p), w_1^-(p), w_{r,-2}(p), w_{r,2}(p)$ of eigenvectors of the hermitian matrix $B(p)$. It is easily seen that (16) transforms the operator B , regarded as an operator in $L^2(\mathbb{R}^3, \mathbb{C}^4)$, into the orthogonal direct sum of four multiplication operators on the measure space. Two first components of this direct sum are the multiplication operators by the constant function equal to unity everywhere, the next direct summand is the operator of multiplication by $\frac{1}{2}r^{-3}$ and the third orthogonal direct summand is the multiplication operator by $\frac{1}{2}r$ (recall that r is the following function: $r(\mathbf{p}) = |\mathbf{p}| = \sqrt{\mathbf{p} \cdot \mathbf{p}}$). Therefore the operator B treated as an operator in \mathcal{H}' is likewise unitarily equivalent to a direct sum of multiplication operators and thus self-adjoint. And similarly B as the operator in \mathcal{H}' has a pure point spectrum $\{1\}$ consisting of just one element 1, and a continuous spectrum equal \mathbb{R}_+ . Indeed any element $\tilde{\varphi} \in \mathcal{H}'$ may be uniquely

⁸Here $p \in \mathcal{O}_{\bar{p}}$ is regarded as the standard function of spatial momentum coordinates \mathbf{p} .

written as the following linear combination

$$\tilde{\varphi}(p) = w_1^+(p) f_+(p) + w_1^-(p) f_-(p) + w_{r-2}(p) f_{0+}(p) + w_{r,2}(p) f_{0-}(p) \quad (17)$$

where f_+, f_-, f_{0+}, f_{0-} are scalar functions on the light cone $\mathcal{O}_{\bar{p}}$. The first two functions f_+, f_- run over the set of all square integrable functions on the light cone $\mathcal{O}_{\bar{p}}$ with respect to the invariant measure $d\mu|_{\mathcal{O}_{\bar{p}}} = \frac{d^3\mathbf{p}}{|\mathbf{p}|}$. The functions f_{0+} range over all functions on $\mathcal{O}_{\bar{p}}$ square integrable with respect to the measure $\frac{d^3\mathbf{p}}{|\mathbf{p}|^3}$, and finally f_{0-} range over all square integrable functions with respect to the measure⁹ $|\mathbf{p}|d^3\mathbf{p}$.

Note that the four elements

$$w_1^+ f_+, w_1^- f_-, w_{r-2} f_{0+}, w_{r,2} f_{0-}$$

of \mathcal{H}' on the right hand side of (17) define orthogonal decomposition \mathcal{H}' into closed invariant subspaces of the self=adjoint operator B , treated as an operator in \mathcal{H}' . Moreover by the formula for the Krein inner-product in $\mathcal{H}', \mathfrak{J}'$ (compare (9) and (10)) the closed subspace spanned by the elements

$$w_{r-2} f_{0+}, w_{r,2} f_{0-},$$

the closed subspace spanned by $w_1^+ f_+$, and the closed subspace spanned by $w_1^- f_-$ are also mutually Krein-orthogonal.

Let \mathcal{H}_{tr} be the closed subspace of the Hilbert space \mathcal{H}' corresponding to the pure point spectrum $\{1\}$ of the operator B in \mathcal{H}' . Then \mathcal{H}_{tr} is spanned by the elements

$$w_1^+ f_+ + w_1^- f_-$$

and the inner product of any two members of \mathcal{H}_{tr} is equal to the Krein-inner product which easily follows from the construction. Thus by construction for every element of \mathcal{H}' existence and uniqueness of the projection on \mathcal{H}_{tr} with respect to the Krein-inner product $(\cdot, \mathfrak{J}'\cdot)$ follows¹⁰.

It is important to understand that the properties of \mathcal{H}_{tr} are of fundamental importance for the construction of the physical space of transversal states and contrary to ordinary Hilbert space the stated above properties of the subspace \mathcal{H}_{tr} are by far not shared by a general (even closed) subspaces of a Krein space.

Because the inner product (\cdot, \cdot) of \mathcal{H}' is just equal to the positive inner product which corresponds through the fundamental symmetry \mathfrak{J}' to the Krein-inner product $(\cdot, \mathfrak{J}'\cdot)$ (in the notation of [6] $(\cdot, \cdot)_{\mathfrak{J}'} = (\cdot, \mathfrak{J}'\mathfrak{J}'\cdot) = (\cdot, \cdot)$) it follows

⁹The measures $\frac{d^3\mathbf{p}}{|\mathbf{p}|^3}$ and $|\mathbf{p}|d^3\mathbf{p}$ are of course not invariant on the cone, but note that the ordinary Hilbert space inner product which they define on \mathcal{H}' is *not* the inner product preserved by the Łopuszński representation. The representation preserves the Krein-inner product.

¹⁰Recall that for a general subspace in a Krein space neither the existence, nor the uniqueness of the projection of a vector on the subspace with respect to the Krein-inner-product is guaranteed. Thus its existence and uniqueness as well as the existence of the corresponding Krein-selfadjoint idempotent need to be proved.

that the subspace \mathcal{H}_{tr} is uniformly positive in the sense of [6], V.5. Being a closed subspace \mathcal{H}_{tr} is regular in the sense of [6], therefore by [6], Ch. V. the subspace \mathcal{H}_{tr} is orthocomplemented with respect to the Krein-inner-product $(\cdot, \mathfrak{J}'\cdot)$ and admits unique projection on \mathcal{H}_{tr} with respect to the Krein-inner-product, which is bounded (boundedness, closedness, continuity always refer to the ordinary Hilbert space inner product of \mathcal{H}' or in general to the corresponding Hilbert space). Thus by [6] there exist bounded Krein-selfadjoint idempotent P (i.e. $P^2 = P$, $P^\dagger = P$, where $P^\dagger = \mathfrak{J}'P^*\mathfrak{J}'$ with the ordinary adjoint P^* in the Hilbert space \mathcal{H}') with range $P\mathcal{H}' = \mathcal{H}_{\text{tr}}$.

Now we define the elements of \mathcal{H}_{tr} as the physical transversal states. But it turns out that in order to account for the Lorentz covariance and the gauge freedom we cannot stay within \mathcal{H}_{tr} . The Łopuszański representation and the representation conjugate to it, whenever applied to a vector $\tilde{\varphi}$ of \mathcal{H}_{tr} , in general transform it into a vector $\tilde{\varphi}''$ which does not lie in \mathcal{H}_{tr} . But the amazing property of these representations is that always

$$\tilde{\varphi}'' = \tilde{\varphi}' + \tilde{\varphi}_0 \quad (18)$$

for a unique vector $\tilde{\varphi}' \in \mathcal{H}_{\text{tr}}$ and a unique $\tilde{\varphi}_0$ whose Krein-inner-product norm vanishes:

$$(\tilde{\varphi}_0, \mathfrak{J}'\tilde{\varphi}_0) = 0,$$

where (\cdot, \cdot) is the inner product in \mathcal{H}' , and which is Krein-orthogonal to \mathcal{H}_{tr} :

$$(\tilde{\varphi}_0, \mathfrak{J}'\tilde{\varphi}''') = 0, \quad \tilde{\varphi}''' \in \mathcal{H}_{\text{tr}}$$

(both $\tilde{\varphi}'$ and $\tilde{\varphi}_0$ in general depend on $\tilde{\varphi}$ and on the applied transformation). Because the Krein-norm of $\tilde{\varphi}'' = \tilde{\varphi}' + \tilde{\varphi}_0$ is equal to the Krein norm of $\tilde{\varphi}'$, and the Krein inner product on \mathcal{H}_{tr} coincides with the ordinary inner product on \mathcal{H}' , and the representations are Krein-isometric, then it follows that the transformation $\tilde{\varphi} \mapsto \tilde{\varphi}'$ which they generate on \mathcal{H}_{tr} is isometric with respect to the ordinary Hilbert space-inner product induced on \mathcal{H}_{tr} by the Krein inner product.

Moreover by construction of the dense core domain \mathfrak{D} of the induced representation (compare [34], Sect. 2) to which the Łopuszański representation is equivalent shows that \mathfrak{D} is likewise dense in the subspace \mathcal{H}_{tr} . It is easily seen because in our case \mathfrak{D} consists of all those $\tilde{\varphi} \in \mathcal{H}'$ which are continuous functions on the cone with compact support, and all of them when projected on \mathcal{H}_{tr} include all the functions of the form

$$w_1^+ f_+ + w_1^- f_-$$

with f_+, f_- continuous of compact support, which are obviously dense in \mathcal{H}_{tr} . Therefore the representations generated by the action modulo unphysical states by the Krein representation (and its conjugation) on the transversal subspace \mathcal{H}_{tr} is not only Hilbert-space isometric but can be uniquely extended to an ordinary unitary representation on \mathcal{H}_{tr} . This is really amazing in view of the quite singular character of the Łopuszański representation (and its conjugation)

for which representor of any boost is unbounded (with respect to the Hilbert space norm of \mathcal{H}'). Now we show that the the Łopuszański representation $WU^{(1,0,0,1)}\mathbf{L}W^{-1}$ and its conjugation $\mathfrak{J}'WU^{(1,0,0,1)}\mathbf{L}W^{-1}\mathfrak{J}'$ does have the property (18). In fact during the proof we give explicit construction of the unitary representation generated on the physical subspace \mathcal{H}_{tr} .

Indeed, in order to show (18) it is sufficient to compute $\Lambda(\alpha^{-1})w_1^+(\Lambda(\alpha)p)$ and $\Lambda(\alpha^{-1})w_1^-(\Lambda(\alpha)p)$, for $\alpha \in SL(2, \mathbb{C})$. It turns out that in general we have

$$\begin{aligned}\Lambda(\alpha^{-1})w_1^+(\Lambda(\alpha)p) &= \Theta_+^+(\alpha, p)w_1^+(p) + \Theta_+^-(\alpha, p)w_1^-(p) + u^+(\alpha, p), \\ \Lambda(\alpha^{-1})w_1^-(\Lambda(\alpha)p) &= \Theta_+^-(\alpha, p)w_1^+(p) + \Theta_+^+(\alpha, p)w_1^-(p) + u^-(\alpha, p),\end{aligned}\quad (19)$$

where the scalar real functions $\Theta_+^+, \Theta_+^-, \Theta_-^+, \Theta_-^-$, have the properties

$$\Theta_-^+ = -\Theta_+^-, \quad \Theta_+^+ = \Theta_-^-$$

and

$$(\Theta_+^+)^2 + (\Theta_+^-)^2 = 1;$$

and where $u^+(\alpha, p)$ and $u^-(\alpha, p)$ are at each point p of the cone $\mathcal{O}_{\bar{p}}$ Krein-orthogonal to both $w_1^+(p)$ and $w_1^-(p)$ and have the Krein-norm zero in the Krein space $\mathbb{C}^4, \mathfrak{J}_{\bar{p}}$:

$$\begin{aligned}(u^+(\alpha, p), \mathfrak{J}_{\bar{p}}w_1^+(p))_{\mathbb{C}^4} &= 0, (u^+(\alpha, p), \mathfrak{J}_{\bar{p}}w_1^-(p))_{\mathbb{C}^4} = 0, (u^+(\alpha, p), \mathfrak{J}_{\bar{p}}u^+(\alpha, p))_{\mathbb{C}^4} = 0, \\ (u^-(\alpha, p), \mathfrak{J}_{\bar{p}}w_1^+(p))_{\mathbb{C}^4} &= 0, (u^-(\alpha, p), \mathfrak{J}_{\bar{p}}w_1^-(p))_{\mathbb{C}^4} = 0, (u^-(\alpha, p), \mathfrak{J}_{\bar{p}}u^-(\alpha, p))_{\mathbb{C}^4} = 0.\end{aligned}$$

In view of the properties (9) and (10) of the Krein-inner product in the Krein space $(\mathcal{H}', \mathfrak{J}')$ of the representation $WU^{(1,0,0,1)}\mathbf{L}W^{-1}$ and on account of the form (6) of the representation, this implies in particular the property (18) of the representation and in fact gives the concrete shape of the unitary representation generated by its action modulo unphysical (of zero Krein norm and zero projection on \mathcal{H}_{tr}) states on the closed subspace \mathcal{H}_{tr} .

Note that the action modulo unphysical states of the conjugate representation $\mathfrak{J}'WU^{(1,0,0,1)}\mathbf{L}W^{-1}\mathfrak{J}'$ generates numerically identical unitary representation on \mathcal{H}_{tr} . This is because the fundamental symmetry \mathfrak{J}' acts as the identity operator on \mathcal{H}_{tr} . Indeed because the decomposition components \mathfrak{J}'_p of \mathfrak{J}' in the direct integral decomposition of \mathfrak{J}' have the form (compare Appendix) $\mathfrak{J}'_p = V(\beta(p)^{-1})\mathfrak{J}_{\bar{p}}V(\beta(p)) = \mathfrak{J}_{\bar{p}}V(\beta(p))^*V(\beta(p)) = \mathfrak{J}_{\bar{p}}B(p)$, then we have

$$\begin{aligned}\mathfrak{J}'_pw_1^+(p) &= w_1^+, \\ \mathfrak{J}'_pw_1^-(p) &= w_1^-, \\ \mathfrak{J}'_pw_{r-2}(p) &= -w_{r2}, \\ \mathfrak{J}'_pw_{r2}(p) &= -w_{r-2}.\end{aligned}$$

From this and from (19) it then easily follows

$$\begin{aligned}\mathfrak{J}'_p\Lambda(\alpha^{-1})\mathfrak{J}'_{\Lambda(\alpha)p}w_1^+(\Lambda(\alpha)p) &= \Theta_+^+(\alpha, p)w_1^+(p) + \Theta_+^-(\alpha, p)w_1^-(p) + \mathfrak{J}'_pu^+(\alpha, p), \\ \mathfrak{J}'_p\Lambda(\alpha^{-1})\mathfrak{J}'_{\Lambda(\alpha)p}w_1^-(\Lambda(\alpha)p) &= \Theta_+^-(\alpha, p)w_1^+(p) + \Theta_+^+(\alpha, p)w_1^-(p) + \mathfrak{J}'_pu^-(\alpha, p).\end{aligned}\quad (20)$$

Now if $u \in \mathcal{H}_p = \mathbb{C}^4$ is any Krein null vector in $(\mathbb{C}^4, \mathfrak{J}_{\bar{p}})$, Krein orthogonal to $w_1^+(p)$ and $w_1^-(p)$ in $(\mathbb{C}^4, \mathfrak{J}_{\bar{p}})$ for every p on the cone, as is the case for both $u^+(\alpha, p), u^-(\alpha, p)$, then u is of the form

$$u = aw_{r^{-2}}^+ + bw_{r^2}^+$$

at each point p (and a, b depending on p). Because $w_{r^{-2}}^+, w_{r^2}^+$ are themselves null in $(\mathbb{C}^4, \mathfrak{J}_{\bar{p}})$ for each p , then u has to be of the form

$$u = aw_{r^{-2}}^+ \text{ or } u = bw_{r^2}^+.$$

Thus

$$\mathfrak{J}'_p u = -aw_{r^2} \text{ or } \mathfrak{J}'_p u = -bw_{r^{-2}},$$

and in any case the vector $\mathfrak{J}'_p u$ is likewise Krein-null and Krein orthogonal to both w_1^+, w_1^- at each point p of the light cone in the Krein space $(\mathbb{C}^4, \mathfrak{J}_{\bar{p}})$. Therefore we can rewrite (20) in the form

$$\begin{aligned} \mathfrak{J}'_p \Lambda(\alpha^{-1}) \mathfrak{J}'_{\Lambda(\alpha)p} w_1^+(\Lambda(\alpha)p) &= \Theta_+^+(\alpha, p) w_1^+(p) + \Theta_-^+(\alpha, p) w_1^-(p) + u_*^+(\alpha, p), \\ \mathfrak{J}'_p \Lambda(\alpha^{-1}) \mathfrak{J}'_{\Lambda(\alpha)p} w_1^-(\Lambda(\alpha)p) &= \Theta_+^-(\alpha, p) w_1^+(p) + \Theta_-^-(\alpha, p) w_1^-(p) + u_*^-(\alpha, p) \end{aligned} \quad (21)$$

where u_*^+, u_*^- are Krein-null and Krein orthogonal to both w_1^+, w_1^- at each point p of the light cone in the Krein space $(\mathbb{C}^4, \mathfrak{J}_{\bar{p}})$. Thus our statement follows and the conjugate Łopuszański representation generates numerically the same unitary representation on \mathcal{H}_{tr} as does the Łopuszański representation itself. It is therefore sufficient to compute $\Theta_+^+, \Theta_-^+, \Theta_+^-, \Theta_-^-$ for the Łopuszański representation as for the conjugate representation they are identical.

It is easily seen that the unitary representation \mathbb{U}, \mathbb{T} (\mathbb{U} stands for representors of the $SL(2, \mathbb{C})$ subgroup, and \mathbb{T} stands for the representors of the translation subgroup) generated on \mathcal{H}_{tr} has the form

$$\begin{aligned} \mathbb{U}(\alpha) \begin{pmatrix} f_+ \\ f_- \end{pmatrix} (p) &= \begin{pmatrix} \Theta_+^+(\alpha, p) & \Theta_-^+(\alpha, p) \\ \Theta_+^-(\alpha, p) & \Theta_-^-(\alpha, p) \end{pmatrix} \begin{pmatrix} f_+(\Lambda(\alpha)p) \\ f_-(\Lambda(\alpha)p) \end{pmatrix}, \\ \mathbb{T}(a) \begin{pmatrix} f_+ \\ f_- \end{pmatrix} (p) &= e^{ia \cdot p} \begin{pmatrix} f_+(p) \\ f_-(p) \end{pmatrix}. \end{aligned}$$

In view of the indicated above properties of the functions $\Theta_+^+, \Theta_-^+, \Theta_+^-, \Theta_-^-$ there exists such a phase $\Theta(\alpha, p)$ that $\Theta_+^+(\alpha, p) = \Theta_-^-(\alpha, p) = \cos \Theta(\alpha, p)$, $\Theta_+^-(\alpha, p) = -\Theta_-^+(\alpha, p) = \sin \Theta(\alpha, p)$, so that the unitary representors $\mathbb{U}(\alpha)$ of the subgroup $SL(2, \mathbb{C})$ of the unitary representation generated on \mathcal{H}_{tr} may be written as

$$\mathbb{U}(\alpha) \begin{pmatrix} f_+ \\ f_- \end{pmatrix} (p) = \begin{pmatrix} \cos \Theta(\alpha, p) & \sin \Theta(\alpha, p) \\ -\sin \Theta(\alpha, p) & \cos \Theta(\alpha, p) \end{pmatrix} \begin{pmatrix} f_+(\Lambda(\alpha)p) \\ f_-(\Lambda(\alpha)p) \end{pmatrix},$$

and recall that (f_+, f_-) compose the Hilbert space of all pairs of functions on the cone which are square integrable with respect to the invariant measure on

the cone. Applying to this Hilbert space and to the unitary representation \mathbb{U}, \mathbb{T} in \mathcal{H}_{tr} (which is naturally identified with the Hilbert space of pairs of functions (f_+, f_-) square integrable on the light cone with the invariant measure) the unitary transformation \mathcal{U} defined by

$$\mathcal{U} \begin{pmatrix} f_1 \\ f_{-1} \end{pmatrix} (p) = \begin{pmatrix} \frac{-i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} f_+(p) \\ f_-(p) \end{pmatrix},$$

we obtain

$$\mathcal{U}^{-1} \mathbb{U}(\alpha) \mathcal{U} \begin{pmatrix} f_1 \\ f_{-1} \end{pmatrix} (p) = \begin{pmatrix} e^{i\Theta(\alpha, p)} & 0 \\ 0 & e^{-i\Theta(\alpha, p)} \end{pmatrix} \begin{pmatrix} f_1(\Lambda(\alpha)p) \\ f_{-1}(\Lambda(\alpha)p) \end{pmatrix}, \quad (22)$$

$$\mathcal{U}^{-1} \mathbb{T}(a) \mathcal{U} \begin{pmatrix} f_+ \\ f_- \end{pmatrix} (p) = e^{ia \cdot p} \begin{pmatrix} f_1(p) \\ f_{-1}(p) \end{pmatrix}, \quad (23)$$

which is precisely the representation of the single-photon states described in [3], §4.3 (formulas (4.23) and (4.22)). For the connection of this representation with the Riemann-Silberstein vector as well as for its construction based on the correspondence with energy-momentum of the classical field and the Staruszkiewicz affine connection on the light cone [32], we refer to [3].

Concerning the concrete form of the phase Θ , or equivalently of the functions $\Theta_+^+, \dots, \Theta_-^-$ in (19), it is essentially a matter of simple computation which follows from the rule (19) and the concrete form of the eigenvectors $w_1^+(p)$, $w_1^-(p)$, $w_{r-2}(p)$, $w_{r-2}(p)$. Because the general formula is quite complicated and the computational recipe given here is simple, we leave to the reader the computation of the general formula for Θ . Here we give only several examples, which nonetheless are sufficient for the proof of (19) and for the reconstruction of the general formula for Θ . To this end let $\alpha_{\mu\nu}$ denote the element of $SL(2, \mathbb{C})$ which corresponds through the natural homomorphism Λ to the rotation $\Lambda(\alpha_{\mu\nu})$ in the $\mu - \nu$ plane. In particular $\Lambda(\alpha_{03})$ denotes the Lorentz rotation in the $0 - 3$ plane and $\Lambda(\alpha_{23})$ stands for the ordinary spatial rotation in the $2 - 3$ plane, i.e. around the first axis. We agree to use λ to denote the hyperbolic angle of the Lorentz rotations $\Lambda(\alpha_{0k})$, and let θ be the angle of the spatial rotations $\Lambda(\alpha_{ik})$. Then in particular we have

$$\begin{aligned} \Lambda(\alpha_{03}^{-1}) w_1^+(\Lambda(\alpha_{03})p) &= w_1^+(p), \\ \Lambda(\alpha_{03}^{-1}) w_1^-(\Lambda(\alpha_{03})p) &= w_1^-(p) + \sinh \lambda \frac{\sqrt{(p^1)^2 + (p^2)^2}}{p^0 \cosh \lambda + p^3 \sinh \lambda} w_{r-2}; \end{aligned}$$

$$\begin{aligned} \Lambda(\alpha_{12}^{-1}) w_1^+(\Lambda(\alpha_{12})p) &= w_1^+(p), \\ \Lambda(\alpha_{12}^{-1}) w_1^-(\Lambda(\alpha_{12})p) &= w_1^-(p); \end{aligned}$$

$$\begin{aligned} \Lambda(\alpha_{23}^{-1})w_1^+(\Lambda(\alpha_{23})p) = \\ \frac{1}{\sqrt{(p^1)^2 + (p^2 \cos \theta + p^3 \sin \theta)^2}} \left(\frac{p^2 p^3 \sin \theta}{\sqrt{(p^1)^2 + (p^2)^2}} + \sqrt{(p^1)^2 + (p^2)^2} \cos \theta \right) w_1^+(p) \\ + \frac{rp^1 \sin \theta}{\sqrt{(p^1)^2 + (p^2 \cos \theta + p^3 \sin \theta)^2} \sqrt{(p^1)^2 + (p^2)^2}} w_1^-(p), \end{aligned}$$

$$\begin{aligned} \Lambda(\alpha_{23}^{-1})w_1^-(\Lambda(\alpha_{23})p) = \\ \frac{-rp^1 \sin \theta}{\sqrt{(p^1)^2 + (p^2 \cos \theta + p^3 \sin \theta)^2} \sqrt{(p^1)^2 + (p^2)^2}} w_1^+(p) \\ + \frac{1}{\sqrt{(p^1)^2 + (p^2 \cos \theta + p^3 \sin \theta)^2}} \left(\frac{p^2 p^3 \sin \theta}{\sqrt{(p^1)^2 + (p^2)^2}} + \sqrt{(p^1)^2 + (p^2)^2} \cos \theta \right) w_1^-(p). \end{aligned}$$

The formulas for $\Lambda(\alpha_{13}^{-1})w_1^+(\Lambda(\alpha_{13})p)$ and $\Lambda(\alpha_{13}^{-1})w_1^-(\Lambda(\alpha_{13})p)$ are obtained by interchanig p^1 and p^2 with each other in the last two formulas respectively.

From the the first two of the formulas just written it is easily seen that the representors of the Lorentz rotations $\Lambda(\alpha_{03})$ in the Łopuszański representation are unbounded, as their action on any element

$$w_1^+ f_+ + w_1^- f_-$$

of \mathcal{H}_{tr} equals

$$\begin{aligned} p \mapsto w_1^+(p) f_+(\Lambda(\alpha_{03})p) + w_1^-(p) f_-(\Lambda(\alpha_{03})p) \\ + \sinh \lambda \frac{\sqrt{(p^1)^2 + (p^2)^2}}{p^0 \cosh \lambda + p^3 \sinh \lambda} w_{r^{-2}}(p) f_-(\Lambda(\alpha_{03})p); \end{aligned}$$

and on the other hand the function $p \mapsto f_-(p)$ being square integrable with respect to the invariant measure $r^{-1}d^3\mathbf{p}$ gives the function $p \mapsto f_-(\Lambda(\alpha_{03})p)$ which is likewise square integrable with respect to the invariant measure on the cone. But in general such a function is not square integrable with respect to the measure $r^{-3}d^3\mathbf{p}$ on the cone.

In general only the Lorentz rotations (“boosts”) produce the additional unphysical Krein zero vector u Krein orthogonal to \mathcal{H}_{tr} when acting on \mathcal{H}_{tr} . Spatial rotations (and of course translations) transform unitarily \mathcal{H}_{tr} into itself.

From the above formulas it follows for example that

$$\begin{aligned} \sin \Theta(\alpha_{03}, p) &= 0, \quad \cos \Theta(\alpha_{03}, p) = 1, \\ \sin \Theta(\alpha_{12}, p) &= 0, \quad \cos \Theta(\alpha_{12}, p) = 1, \\ \sin \Theta(\alpha_{23}, p) &= \frac{rp^1 \sin \theta}{\sqrt{(p^1)^2 + (p^2 \cos \theta + p^3 \sin \theta)^2} \sqrt{(p^1)^2 + (p^2)^2}}; \\ \cos \Theta(\alpha_{23}, p) &= \frac{1}{\sqrt{(p^1)^2 + (p^2 \cos \theta + p^3 \sin \theta)^2}} \left(\frac{p^2 p^3 \sin \theta}{\sqrt{(p^1)^2 + (p^2)^2}} + \sqrt{(p^1)^2 + (p^2)^2} \cos \theta \right). \end{aligned}$$

The formulas for $\sin \Theta(\alpha_{13}, p)$ and $\cos \Theta(\alpha_{13}, p)$ are obtained by interchanging p^1 and p^2 with each other in the last two formulas respectively.

The representation given by (22) and (23) has already the form of direct sum of unitary representations concentrated on the single (zero mass) orbit $\mathcal{O}_{(1,0,0,1)} \cong SL(2, \mathbb{C})/G_{(1,0,0,1)}$, with the opposite signs of helicities of the representations of the small group $G_{(1,0,0,1)}$ (double cover of the Euclidean group). Because Θ depends on $\alpha \in SL(2, \mathbb{C})$ only through the natural homomorphism Λ of the $SL(2, \mathbb{C})$ onto the Lorentz group, then it is easily seen that the helicities must be integer numbers (and not half-integer), and thus giving true (single valued) representations of the Poincaré group (and not the double valued). A simple computation shows that (22) and (23) is unitarily equivalent to the direct sum of unitary representations induced respectively by the irreducible helicity 1 and -1 representations of the small subgroup $G_{(1,0,0,1)}$. Thus according to the Mackey theory of induced representations (22) and (23) is equivalent to $[m = 0, h = 1] \oplus [m = 0, h = -1]$.

Apparently we could restrict to the subspace \mathcal{H}_{tr} and to the unitary representation (22) and (23) acting in \mathcal{H}_{tr} , forgetting about the surrounding Krein space and the Łopuszański representation and its conjugation. But although the representation (22) and (23) is localizable in the sense of Jauch and Piron [18] (for the proof compare [1]) it is not equivalent to any representation which in the momentum picture has the multiplier independent of $p \in \mathcal{O}_{\bar{p}}$, i.e. the transformation it generates in the position picture is nonlocal. It is really amazing that in passing from the vector potential to the Riemann-Silberstein vector (or to the electric and magnetic field) the p -dependence of the multiplier of the representation (22) and (23) is compensated for by the p -dependence of the polarization vector, so that the application of the second quantization functor Γ to the representation (22) and (23) gives indeed the quantum Riemann-Silberstein (or quantum electric and magnetic) field with local transformation formula, compare [3]. Unfortunately the compensation breaks down for the vector potential itself (the additional factor $p^0(\mathbf{p})$ is crucial when passing from the pair of helicity +1 and -1 vector potentials to the Riemann-Silberstein vector) so that locality of the vector potential is not preserved within this scheme. We need the surrounding Krein space and the Łopuszański representation (and its conjugation) in order to save the local character of the transformation law for the vector potential. Alternatively one can interpret the situation as follows. In order to restore the locality of the inverse Fourier transforms of the transformation of the elements $w_1^+ f_+ + w_1^- f_-$ of \mathcal{H}_{tr} acted on by the representation (\mathbb{U}, \mathbb{T}) one has to additionally apply the appropriate gauge transformation, i.e. one has to add to the inverse Fourier transforms of the transformed element $\mathbb{U}(\alpha)(w_1^+ f_+ + w_1^- f_-)$ a gradient of a function g depending on the initial state (e.g. on the inverse Fourier transform of $w_1^+ f_+ + w_1^- f_-$) highly non-locally, and explicitly on x and α . Accordingly to the results presented above, the p -dependence of the phase Θ is just compensated for by the p dependence of the eigenvectors w_1^+, w_1^- , whenever α corresponds to the spatial rotation, so that the transformation remains local in position picture in this particular case. But for a general Lorentz rotation, u^+, u^- (and Θ) depend on p nontrivially

so that this dependence is not cancelled by the p -dependence of w_1^+, w_1^- and $\mathbb{U}(\alpha)$ in the position picture acts in general non-locally. Indeed, in general the difference between the representations U and \mathbb{U} acting on \mathcal{H}_{tr} is equal

$$U(\alpha)(w_1^+ f_+ + w_1^- f_-) - \mathbb{U}(\alpha)(w_1^+ f_+ + w_1^- f_-) = \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \cdot \tilde{g}(\alpha, f_+, f_-, p),$$

where the scalar function \tilde{g} depends on α, f_+, f_-, p . Therefore in the position picture the difference between the local transform $U(\alpha)$ and the non-local transformation $\mathbb{U}(\alpha)$ is equal to a gradient of a function g , depending on α, f_+, f_- and explicitly on x . The function $U(\alpha)(w_1^+ f_+ + w_1^- f_-)$ belongs to \mathcal{H}' whenever \tilde{g} is square integrable on the cone with respect to the invariant measure. In particular

$$\tilde{g}(\alpha_{03}, f_+, f_-, p) = \sinh \lambda \frac{\sqrt{(p^1)^2 + (p^2)^2}}{p^0 \cosh \lambda + p^3 \sinh \lambda} \frac{1}{\sqrt{2} p^0} f_-(\Lambda(\alpha_{03})p);$$

so that it is easily seen that in passing to the position picture the inverse Fourier transform g of \tilde{g} depend in a non-local way on the initial state (inverse Fourier transform of $w_1^+ f_+ + w_1^- f_-$).

3.1 Appendix: Krein-isometric representations concentrated on single orbits and local wave functions

This Section is based on a generalization of Mackey's theory, presented in [34].

We assume the results of the mentioned generalization and use them in the construction of single particle wave functions which in the position picture have local transformation rule. The novelty lies in the application to Krein-isometric representations in Krein spaces and in that it is more consequently related to the theory of induced representations.

Representations of the double cover $T_4 \otimes SL(2, \mathbb{C})$ of the Poincaré group¹¹ considered here are in general not unitary but Krein-unitary and even only Krein-isometric (for definitions compare Sect. 1 and 2 of [34]) with the properties (1) and (3) motivated by the properties of representations acting in the Krein-Fock spaces of the free fields underlying QED (and the Standard Model). The first property is that the Gupta-Bleuler operator \mathfrak{J} – playing the role of the fundamental symmetry of the Krein space in question, commutes with translations. Consider first a Krein-isometric representation acting in one particle Krein subspace $(\mathcal{H}, \mathfrak{J})$ (or in its subspace) of the Krein-Fock space in question. Because translations (we mean of course their representors) commute with \mathfrak{J} , they are not only Krein-isometric but unitary with respect to the Hilbert space

¹¹We denote the representor of $(a, \alpha) \in T_4 \otimes SL(2, \mathbb{C})$ by $U_{(a, \alpha)}$, and the convention in which the Lorentz transformation $\Lambda(\alpha)$ corresponding to $\alpha \in SL(2, \mathbb{C})$ is an antihomomorphism, and with the right action of Λ on $a \in \widehat{T}_4$.

inner product of the Krein space $(\mathcal{H}, \mathfrak{J})$ in question. Let P^0, \dots, P^3 be the respective generators of the translations (they do exist by the strong continuity assumption posed on the Krein-isometric representation – physicist’s everyday computations involve the generators and thus our assumption is justified, compare Sect. 2 of [34]). Let \mathcal{C} be the commutative C^* -algebra generated by the functions $f(P^0, \dots, P^3)$ of translation generators P^0, \dots, P^3 , where f is continuous on \mathbb{R}^4 and vanishes at infinity. Let

$$\mathcal{H} = \int_{\text{sp}(P^0, \dots, P^3)} \mathcal{H}_p \, d\mu(p) \quad (24)$$

be the direct integral decomposition of \mathcal{H} corresponding to the algebra \mathcal{C} (in the sense of [22] or [31]) with a spectral measure μ on the joint spectrum $\text{sp}(P^0, \dots, P^3)$ of the translation generators. We may identify $\text{sp}(P^0, \dots, P^3)$ with a subset of the group \widehat{T}_4 dual to the translation group T_4 . Moreover we may assume that the algebra \mathcal{C} and the spectral measure corresponding to the above decomposition (24) are of uniform multiplicity, compare Theorem 5, Sect. 5 of [34]. Let us denote the translation representor $U_{(a,1)}$ just by $T(a)$ and the representor $U_{(0,\alpha)}$ of the $SL(2, \mathbb{C})$ subgroup just by $U(\alpha)$. By the multiplication rule in $T_4 \otimes SL(2, \mathbb{C})$ it follows that

$$T(a\Lambda(\alpha^{-1})) = U(\alpha)^{-1}T(a)U(\alpha), \quad (25)$$

such that

$$U(\alpha)^{-1}P^\nu U(\alpha) = \Lambda(\alpha^{-1})^\nu_\mu P^\mu \quad (\text{summation over } \mu)$$

so that $U(\alpha)^{-1}E(S)U(\alpha) = E(\Lambda(\alpha^{-1})S)$ for $S \subset \text{sp}(P^0, \dots, P^3)$, i. e. $U(\alpha)$ acts on the joint spectrum of P^0, \dots, P^3 as the ordinary right action of the Lorentz transformation $\Lambda(\alpha^{-1})$. Moreover we may identify

$\text{sp}(P^0, \dots, P^3) \subset \widehat{T}_4$ with the orbit $\mathcal{O}_{\bar{p}}$ under the standard action of the Lorentz group of a single point $\bar{p} = \bar{p}(m)$ in the vector space \mathbb{R}^4 endowed with the Minkowski pseudo-metric form g_M with the signature $(1, -1, -1, -1)$, and with the invariant measure μ_m on the orbit $\mathcal{O}_{\bar{p}} = \{p : g_M(p, p) = m^2\}$ induced by the invariant Lebesgue measure on \mathbb{R}^4 equal to the Haar measure on \widehat{T}_4 . Because the fundamental symmetry \mathfrak{J} commutes with P^0, \dots, P^3 it is decomposable with respect to the decomposition (24), and let $p \mapsto \mathfrak{J}_p$ be its decomposition with respect to (24), i. e.

$$\mathfrak{J} = \int_{\text{sp}(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \mathfrak{J}_p \, d\mu|_{\mathcal{O}_{\bar{p}}}(p)$$

with \mathfrak{J}_p being a fundamental symmetry in \mathcal{H}_p . Because of the uniform multiplicity $\mathcal{H}_p \cong \mathcal{H}_{p\bar{w}ig}$, $p \in \mathcal{O}_{\bar{p}}$. Moreover every element $\tilde{\psi} \in \mathcal{H}$ may be identified with the function $\mathcal{O}_{\bar{p}} \ni p \mapsto \tilde{\psi}(p) \in \mathcal{H}_{\bar{p}}$ equal to the decomposition of $\tilde{\psi}$ with

respect to (24). Therefore in the notation of von Neumann

$$\tilde{\psi} = \int_{sp(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \tilde{\psi}(p) \sqrt{d\mu|_{\mathcal{O}_{\bar{p}}}}(p).$$

We may assume that \mathfrak{J}_p does not depend on p – which is still sufficient for the representations acting on one-particle states as well as for the decomposition of their tensor products (the latter assertion will be proved in the further stages of this paper). In this Section we identify every $\tilde{\psi} \in \mathcal{H}$ with the corresponding function $p \mapsto \tilde{\psi}(p)$ – its decomposition.

Now for each $\alpha \in SL(2, \mathbb{C})$ let us define the following operator $D(\alpha)$ (compare [28, 35])

$$D(\alpha)\tilde{\psi}(p) = \tilde{\psi}(\Lambda(\alpha)p).$$

By the Lorentz invariance of the measure μ on the orbit $\mathcal{O}_{\bar{p}}$ it follows that $D(\alpha)$ is unitary for every $\alpha \in SL(2, \mathbb{C})$. Moreover, because the components \mathfrak{J}_p in the decomposition of \mathfrak{J} do not depend on $p \in \mathcal{O}_{\bar{p}}$, it easily follows that $D(\alpha)$ commutes with \mathfrak{J} , so that $D(\alpha)$ is Krein-unitary for each $\alpha \in SL(2, \mathbb{C})$. Thus $\alpha \mapsto D(\alpha)$ gives a unitary and Krein-unitary representation of $SL(2, \mathbb{C})$:

$$D(\alpha\beta) = D(\alpha)D(\beta).$$

Let F be any Baire function on $sp(P^0, \dots, P^3) = \mathcal{O}_{\bar{p}}$, and let $F(P) = F(P^0, \dots, P^3)$ be the operator function of P^0, \dots, P^3 , i. e. operator

$$F(P)\tilde{\psi}(p) = F(p)\tilde{\psi}(p).$$

An easy computation shows that

$$D(\alpha)F(P) = F(\Lambda(\alpha)P)D(\alpha), \quad (26)$$

where $F(\Lambda(\alpha)P) = F(\Lambda(\alpha)^\mu_\nu P^\nu)$ (summation with respect to ν). Joining (25) and (26) it follows that

$$[U(\alpha)D(\alpha)^{-1}, T(a)] = 0. \quad (27)$$

Thus $Q(\alpha) = U(\alpha)D(\alpha)^{-1}$ commutes with the elements of the C^* - algebra \mathcal{C} and it is decomposable with respect to (24) (in other words it is a function of the operators P^0, \dots, P^3). Denote the components $Q(\alpha)_p$ of $Q(\alpha)$ with respect to this decomposition just by $Q(\alpha, p)$. Recall that they are operators acting in $\mathcal{H}_{\bar{p}}$, so that

$$Q(\alpha) = \int_{\mathcal{O}_{\bar{p}}} Q(\alpha, p) d\mu|_{\mathcal{O}_{\bar{p}}}(p).$$

Thus in the notation of von Neumann [22]

$$U(\alpha)\tilde{\psi} = Q(\alpha)D(\alpha)\tilde{\psi} = \int_{\mathcal{O}_{\bar{p}}} Q(\alpha, p)(D(\alpha)\tilde{\psi})(p) \sqrt{d\mu|_{\mathcal{O}_{\bar{p}}}}(p),$$

where $p \mapsto (D(\alpha)\tilde{\psi})(p)$ is the decomposition of $D(\alpha)\tilde{\psi}$, so that

$$p \mapsto (U(\alpha)\tilde{\psi})(p) = Q(\alpha, p)(D(\alpha)\tilde{\psi})(p)$$

is the decomposition of $U(\alpha)\tilde{\psi}$.

Because $\alpha \mapsto U(\alpha)$ is a representation it follows that the components $Q(\alpha, p)$ of $Q(\alpha)$ have the following multiplier property

$$Q(\delta\alpha, p) = Q(\delta, p)Q(\alpha, \Lambda(\delta)p), \quad p \in \mathcal{O}_{\bar{p}}, \alpha, \delta \in SL(2, \mathbb{C}).$$

In particular

$$Q(e, p) = 1, \quad Q(\alpha, p)^{-1} = Q(\alpha^{-1}, \Lambda(\alpha)p).$$

If we consider any Krein-isometric operator W which preserves the invariant core domain of the Krein-isometric representation U (i.e. the domain \mathfrak{D} of Sect. 2 of [34]) and which is decomposable with respect to (24) with the decomposition $p \mapsto W(p)$, then (with $\tilde{\Psi} = W\tilde{\psi}$)

$$WU(\alpha)W^{-1}\tilde{\Psi} = \int_{\mathcal{O}_{\bar{p}}} W(p)Q(\alpha, p)W(\Lambda(\alpha)p)^{-1}(D(\alpha)\tilde{\Psi})(p) \sqrt{d\mu|_{\mathcal{O}_{\bar{p}}}}(p) \quad (28)$$

with $WU(\alpha)W^{-1}$ being another Krein-isometric representation, forces

$$Q'(\alpha, p) = W(p)Q(\alpha)W(\Lambda(\alpha)p)^{-1} \quad (29)$$

to be another multiplier:

$$Q'(\delta\alpha, p) = Q'(\delta, p)Q'(\alpha, \Lambda(\delta)p), \quad p \in \mathcal{O}_{\bar{p}}, \alpha, \beta \in SL(2, \mathbb{C}),$$

corresponding to the representation $\alpha \mapsto WU(\alpha)W^{-1}$.

Moreover the core domain \mathfrak{D} have the following *pervasive*¹² property that there exist a sequence $\{f_l\}_{l \in \mathbb{N}}$ of elements of \mathfrak{D} such that for all $p \in \text{sp}(P^0 \dots, P^3) = \mathcal{O}_{\bar{p}}$ (compare Lemma 6 of [34]) $\{f_l(p)\}_{l \in \mathbb{N}}$ is dense in $\mathcal{H}_{\bar{p}} = \mathcal{H}_p$.

Now the operator $Q(\alpha, p)$ is Krein-unitary for almost all $p \in \mathcal{O}_{\bar{p}}$. Indeed we have

$$Q(\alpha, p)\mathfrak{J}_{\bar{p}}Q(\alpha, p)^*\mathfrak{J}_{\bar{p}}f_l(p) = f_l(p) \quad \text{and} \\ \mathfrak{J}_{\bar{p}}Q(\alpha, p)^*\mathfrak{J}_{\bar{p}}f_l(p)Q(\alpha, p) = f_l(p) \quad p \in \mathcal{O}_{\bar{p}}, l \in \mathbb{N}.$$

Because for each $p \in \mathcal{O}_{\bar{p}}$ $\{f_l(p)\}_{l \in \mathbb{N}}$ is dense in $\mathcal{H}_{\bar{p}}$ and because the representation $\alpha \mapsto U(\alpha)$ is locally finite with respect to the spectral measure E of T determining the corresponding direct integral decomposition (24), i.e. fulfils (3), then

$$Q(\alpha, p)\mathfrak{J}_{\bar{p}}Q(\alpha, p)^*\mathfrak{J}_{\bar{p}} = \mathbf{1} \quad \text{and} \quad \mathfrak{J}_{\bar{p}}Q(\alpha, p)^*\mathfrak{J}_{\bar{p}}Q(\alpha, p) = \mathbf{1},$$

and $Q(\alpha, p)$ is Krein-unitary for all $p \in \mathcal{O}_{\bar{p}}$. In fact in case of the single particle representations the restriction T of the representation to translations has finite

¹²Term introduced by Mackey in [21].

uniform multiplicity so that $\mathcal{H}_{\bar{p}}$ has finite dimension, so that the unitarity of $Q(\alpha, p)$

for almost all p immediately follows independently of the assumption (3) in this case.

It is well known that each element $p = (p^0, \dots, p^3) \in \mathbb{R}^4$ of the dual group $\widehat{T}_4 \supset \text{sp}(P^0, \dots, P^3)$ may be represented by the hermitean 2×2 matrix $\hat{p} = p^0 \mathbf{1} + p^1 \sigma_1 + p^2 \sigma_2 + p^3 \sigma_3$, where σ_i are the Pauli matrices, and with the action of the Lorentz transformation $\Lambda(\alpha)p$ on p given by $\alpha \hat{p} \alpha^* = \widehat{\Lambda(\alpha^{-1})p}$. Now let \bar{p} be any fixed point of the orbit $\mathcal{O}_{\bar{p}}$. Now we associate bi-uniquely an element $\beta(p) \in SL(2, \mathbb{C})$ with every $p \in \mathcal{O}_{\bar{p}}$ such that $\beta(p)^{-1} \widehat{\bar{p}} \beta(p)^{* -1} = \hat{p}$, i. e. $\Lambda(\beta(p))\bar{p} = p$ and $\Lambda(\beta(p)^{-1})p = \bar{p}$. Of course the function $p \mapsto \beta(p) = \beta_{\bar{p}}(p)$ depends on the orbit $\mathcal{O}_{\bar{p}}$, but we discard the subscript \bar{p} at $\beta(p)$ in order to simplify notation, as in the most part of this Sect. we are concerned with a fixed orbit.

It follows that $\gamma(\alpha, p) = \beta(p)\alpha\beta(\Lambda(\alpha)p)^{-1}$ is an element of the subgroup $G_{\bar{p}}$ stationary for \bar{p} : $\Lambda(\gamma(\alpha, p))\bar{p} = \bar{p}$, or $\gamma(\alpha, p)\widehat{\bar{p}}\gamma(\alpha, p)^* = \widehat{\bar{p}}$. Therefore every $\alpha \in SL(2, \mathbb{C})$ has the following factorization:

$$\alpha = \beta(p)^{-1} \gamma(\alpha, p) \beta(\Lambda(\alpha)p).$$

Thus because $Q(\alpha, p)$ is a multiplier we obtain

$$\begin{aligned} Q(\alpha, p) &= Q(\beta(p)^{-1} \gamma(\alpha, p) \beta(\Lambda(\alpha)p, p)) \\ &= Q(\beta(p)^{-1}, p) Q(\gamma(\alpha, p), \bar{p}) Q(\beta(\Lambda(\alpha)p), \bar{p}). \end{aligned} \quad (30)$$

Now let us introduce the operator W decomposable with respect to (24) whose decomposition function is given by

$$p \mapsto W(p) = Q(\beta(p), \bar{p}). \quad (31)$$

Because the components $Q(\alpha, p)$ of $Q(\alpha)$ compose a multiplier, then $W(p)^{-1} = Q(\beta(p)^{-1}, p)$, so that the operator W^{-1} has the decomposition $p \mapsto W(p)^{-1} = Q(\beta(p)^{-1}, p)$. By construction W is a Krein-isometric operator which preserves the core domain \mathfrak{D} of the initial representation U and moreover by (30) we have:

$$Q(\alpha, p) = W(p)^{-1} Q(\gamma(\alpha, p), \bar{p}) W(\Lambda(\alpha)p).$$

Comparison with (28) and (29) shows that the original Krein-isometric representation U is equivalent to the Krein-isometric representation $W^{-1}UW$, where¹³

$$p \mapsto W^{-1}U(\alpha)W\tilde{\psi}(p) = Q(\gamma(\alpha, p), \bar{p})D(\alpha)\tilde{\psi}(p) = Q(\gamma(\alpha, p), \bar{p})\tilde{\psi}(\Lambda(\alpha)p)$$

is the decomposition of $W^{-1}U(\alpha)W\tilde{\psi}$. Note that for γ, γ' ranging over the subgroup $G_{\bar{p}}$ stationary for \bar{p} we have

$$Q(\gamma\gamma', \bar{p}) = Q(\gamma, \bar{p})Q(\gamma', \bar{p}),$$

¹³We have denoted $\tilde{\psi}$ and $W\tilde{\psi}$ by the same letter $\tilde{\psi}$, we hope this will not cause any misunderstanding.

so that $\gamma \mapsto Q(\gamma, \bar{p})$ is a Krein-unitary representation of the subgroup $G_{\bar{p}}$ of $SL(2, \mathbb{C})$ stationary for \bar{p} . Thus the initial representation is equivalent to the representation (we denote it by the same letters U, T as the initial one) whose action on the decomposition functions is given by the following formula

$$\begin{aligned} U(\alpha)\tilde{\psi}(p) &= Q(\gamma(\alpha, p), \bar{p})\tilde{\psi}(\Lambda(\alpha)p), \\ T(a)\tilde{\psi}(p) &= e^{ia \cdot p}\tilde{\psi}(p) = e^{ig_M(a, p)}\tilde{\psi}(p), \end{aligned} \quad (32)$$

where $\gamma \mapsto Q(\gamma, \bar{p})$ is a Krein-unitary representation of the subgroup $G_{\bar{p}}$ stationary for a fixed point \bar{p} belonging to the orbit $\mathcal{O}_{\bar{p}} = \text{sp}(P^0, \dots, P^3)$.

Our next step is to find an explicit formula for the unitary and Krein-unitary transformation (we denote it likewise by W) which applied to vector states $\tilde{\psi}$ of the representation $U_{(a,1)} = T(a)$, $U_{(0,\alpha)} = U(\alpha)$ with $T(a)$ and $U(\alpha)$ given by (32) gives a transformation formula with a multiplier independent of $p \in \mathcal{O}_{\bar{p}}$, i. e. W is such that the Fourier transform

$$\varphi(x) = (2\pi)^{-3/2} \int_{\mathcal{O}_{\bar{p}}} \tilde{\varphi}(p) e^{-ip \cdot x} d\mu|_{\mathcal{O}_{\bar{p}}}(p) \quad (33)$$

of $\tilde{\varphi} = W\tilde{\psi}$ has a local transformation law, where $d\mu(p)$ is the invariant measure induced on the orbit $\mathcal{O}_{\bar{p}}$ by the Lebesgue measure on \mathbb{R}^4 .

To this end we need a representation $\alpha \mapsto V(\alpha)$ of $SL(2, \mathbb{C})$ acting in the Krein space $(\mathcal{H}_{\bar{p}}, \mathfrak{J}_{\bar{p}})$ which extends the Krein-unitary representation $\gamma \mapsto Q(\gamma, \bar{p})$ of the subgroup $G_{\bar{p}} \subset SL(2, \mathbb{C})$ to a representation of the whole $SL(2, \mathbb{C})$ group. V need not be Krein-unitary (resp. unitary in case $\mathfrak{J}_{\bar{p}} = \mathbf{1}$) It turns out that such extensions V do exist for the Krein-unitary (resp. unitary in case $\mathfrak{J}_{\bar{p}} = \mathbf{1}$) representations $\gamma \mapsto Q(\gamma, \bar{p})$ associated with the representations concentrated on single orbits which arise in the process of decomposition of the tensor products of representations acting in single particle subspaces. For example they are well known for the representations $\gamma \mapsto Q(\gamma, \bar{p})$ associated with the representations concentrated on single orbits which arise in decompositions of tensor products of spin one-half, non-zero-mass representations (in this case $\mathfrak{J}_{\bar{p}} = \mathbf{1}$ and the representations $\gamma \mapsto Q(\gamma, \bar{p})$ are unitary).

Namely let us define the transformation W whose action on decomposition functions is defined in the following manner

$$\tilde{\varphi}(p) = W\tilde{\psi}(p) = V(\beta(p)^{-1})\tilde{\psi}(p). \quad (34)$$

Then we have

$$\begin{aligned} WU(\alpha)W^{-1}\tilde{\varphi}(p) &= V(\beta(p)^{-1})V(\gamma(\alpha, p))V(\beta(\Lambda(\alpha)p))\tilde{\varphi}(\Lambda(\alpha)p) \\ &= V\left(\beta(p)^{-1}\beta(p)\alpha\beta(\Lambda(\alpha)p)^{-1}\beta(\Lambda(\alpha)p)\right)\tilde{\varphi}(\Lambda(\alpha)p) = V(\alpha)\tilde{\varphi}(\Lambda(\alpha)p), \end{aligned}$$

therefore

$$WU(\alpha)W^{-1}\tilde{\varphi}(p) = V(\alpha)\tilde{\varphi}(\Lambda(\alpha)p), \quad (35)$$

such that the Fourier transform φ defined by (33) of $\tilde{\varphi}$ has a local transformation formula

$$U(\alpha)\varphi(x) = V(\alpha)\varphi(x\Lambda(\alpha^{-1})) = V(\alpha)\varphi(x_\nu\Lambda(\alpha^{-1})_\mu^\nu) \quad (36)$$

(summation with respect to ν), where we have used again the symbol U for the representation in the space of Fourier transforms φ hoping that it will not cause any serious misunderstandings.

Let

$$\left(\mathcal{H} = \int_{sp(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \mathcal{H}_{\bar{p}} d\mu|_{\mathcal{O}_{\bar{p}}}(p), \quad \mathfrak{J} = \int_{sp(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \mathfrak{J}_{\bar{p}} d\mu|_{\mathcal{O}_{\bar{p}}}(p) \right)$$

be the Krein space of the representation (32), which we may assume to be equal to the Krein space of the initial representation, as the transformation W given by (31) preserves the core set \mathfrak{D} of the initial representation and is Krein-isometric.

Let

$$\left(\mathcal{H}' = \int_{sp(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \mathcal{H}'_p d\mu|_{\mathcal{O}_{\bar{p}}}(p), \quad \mathfrak{J}' = \int_{sp(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \mathfrak{J}'_p d\mu|_{\mathcal{O}_{\bar{p}}}(p) \right),$$

be the Krein space with the Hilbert space inner product in \mathcal{H}' defined by

$$(\tilde{\varphi}, \tilde{\varphi}') = \int_{sp(P^0, \dots, P^3) \cong \mathcal{O}_{\bar{p}}} \left(\tilde{\varphi}(p), \tilde{\varphi}'(p) \right)_p d\mu|_{\mathcal{O}_{\bar{p}}}(p),$$

where

$$\left(\tilde{\varphi}(p), \tilde{\varphi}'(p) \right)_p = \left(\tilde{\varphi}(p), V(\beta(p))^* V(\beta(p)) \tilde{\varphi}'(p) \right)_{\mathcal{H}_{\bar{p}}}, \quad \tilde{\varphi}(p), \tilde{\varphi}'(p) \in \mathcal{H}'_p,$$

with the inner product $(\cdot, \cdot)_{\mathcal{H}_{\bar{p}}}$ of the Hilbert space $\mathcal{H}_{\bar{p}}$ (with the convention assumed here, that it is conjugate linear in the first variable¹⁴); and let the decomposition components of the fundamental symmetry \mathfrak{J}' defined by

$$\mathfrak{J}'_p = V(\beta(p))^{-1} \mathfrak{J}_{\bar{p}} V(\beta(p)).$$

We then have the following

THEOREM. *The transformation W defined by (34), which transforms $\tilde{\psi}$ belonging to the Krein space $(\mathcal{H}, \mathfrak{J})$ of the initial representation, equal to the Krein space of the representation defined by (32), onto the Krein space $(\mathcal{H}', \mathfrak{J}')$ of elements $\tilde{\varphi}$ of the representation (35), is unitary and Krein-unitary.*

¹⁴This convention is assumed in most of the physical literature.

REMARK. The components \mathfrak{J}'_p of the decomposition of the fundamental symmetry \mathfrak{J}' depend in general on $p \in \mathcal{O}_{\bar{p}}$, because $V(\beta(p))$ – although being Krein-unitary and unitary in the respective Krein space $\mathcal{H}'_p, \mathfrak{J}'_p$ for all p – are not in general unitary in the Hilbert space $\mathcal{H}_{\bar{p}}$:

$$\begin{aligned}\mathfrak{J}'_p &= V(\beta(p)^{-1}) \mathfrak{J}_{\bar{p}} V(\beta(p)) \\ &= \mathfrak{J}_{\bar{p}} V(\beta(p))^* \mathfrak{J}_{\bar{p}} V(\beta(p)) = \mathfrak{J}_{\bar{p}} V(\beta(p))^* V(\beta(p)),\end{aligned}$$

where $\mathfrak{J}_{\bar{p}}$ does not depend on p and $V(\beta(p))^* V(\beta(p))$ depends on p .

Therefore construction of a local single particle wave function is nothing more than the application of the above construction of the transformation $\psi \mapsto \tilde{\varphi}$ with the properties indicated by the above theorem. The crucial point being that the Fourier transform (33) of the transformed $\tilde{\varphi}$ has a local transformation law.

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References

- [1] Amrein, W. O.: Helv. Phys. Acta 42, 149 (1969).
- [2] Bialynicki-Birula, I.: Acta Phys. Polon. 86, 97 (1994).
- [3] Bialynicki-Birula, I.: Progress in Optics 36, 245 (1996); Preprint quant-ph/0508202.
- [4] Blanchard, P., Seneor, R.: Annales de L' I. H. P. A23, 147 (1975).
- [5] Bleuler, K: Helv. Phys. Acta 23, 567 (1950).
- [6] Bogнар, J.: Indefinite Inner Product Spaces. Springer, Berlin (1974).
- [7] Bogoliubov, N. N., Shirkov, D. V.: Introduction to the Theory of Quantized Fields. New York (1959), second ed. John Wiley & Sons, Inc., New York, Chichester, Brisbane, Toronto, 1980.
- [8] Bohm, D.: Quantum theory. Constable, London, 1954, p. 91.
- [9] Bohr, N., Rosenfeld, L.: Mat.-fys. Medd. Dan. Vid. Selsk. 12, bo. 8 (1933).
- [10] Dütsch, M., Krahe, F., Scharf, G.: Nuovo Cimento A 103, 871 (1990).
- [11] Dütsch, M., Krahe, F., Scharf, G.: Nuovo Cimento A 1029, 871 (1993).
- [12] Dütsch, M., Krahe, F., Scharf, G.: Nuovo Cimento A 107, 375 (1994).

- [13] Dütsch, M., Krahe, F., Scharf, G.: *Nuovo Cimento A* 108, 737 (1995).
- [14] Dütsch, M., Fredenhagen, K.: *Commun. Math. Phys.* 203, 71 (1999).
- [15] Epstein, H., Glaser, V.: *Ann. Inst. H. Poincaré A* 19, 211 (1973).
- [16] Epstein, H., Glaser, V.: Contribution to the meeting on renormalization theory. C. N. R. S., Marseille, June 1971; C. E. R. N., preprint TH 1344. (1951).
- [17] Haag, R.: *Local Quantum Physics*. Springer Verlag, 1996.
- [18] Jauch, J., M., Piron, C.: *Helv. Phys. Acta* 40, 559 (1967).
- [19] Łopuszański, J.: *Rachunek spinorów*. PWN, Warszawa 1985.
- [20] Łopuszański, J.: *Fortschritte der Physik* 26, 261 (1978).
- [21] Mackey, G. W.: *Ann. Math.* 55, 101 (1952).
- [22] von Neumann, J.: *Ann. Math.* 50, 401 (1949).
- [23] Neumark, M. A.: *Math. Sb.* 65, 198 (1964).
- [24] Neumark, M. A.: *Math. Ann.* 162, 147 (1965).
- [25] Neumark, M. A.: *Acta. Sci. Szeged* 26, 3 and 201 (1965).
- [26] Neumark, M. A.: *Izv. AN SSSR (Math. Series)* 30, 1111 (1966).
- [27] Neumark, M. A.: *Izv. AN SSSR (Math. Series)* 30, 1229 (1966).
- [28] Ohnuki, Y.: *Unitary representations of the Poincaré group and relativistic wave equations*, World Scientific, Singapore 1988.
- [29] Power, E. A.: *Introductory Quantum Electrodynamics*. Longmans, London, 1964, p. 61.
- [30] Segal, I. E.: *Ann. Math.* 57, 401 (1953).
- [31] Segal, I. E.: *Decomposition of Operator Algebras. I. Memoirs of the American Mathematical Society*. No. 9, 1951.
- [32] Staruszkiewicz, A.: *Acta Phys. Polon.* B4, 57 (1973).
- [33] Streater, R. F. and Wightman, A. S.: *PCT, Spin and Statistics, and All That*, W. A. Benjamin, Inc., New York, 1964.
- [34] Wawrzycki, J.: Preprint math-ph/1504.02273.
- [35] Wigner, E. P.: *Ann. Math.* 40, 149 (1939).
- [36] Woronowicz, S. L.: *Studia Mathematica*, 39, 217, (1971).